

**B.Sc. IV SEMESTER**

**Mathematics**

**PAPER – II**

**GROUP THEORY, FOURIER SERIES  
AND  
DIFFERENTIAL EQUATIONS**

# **UNIT-IV**

## **Differential Equation-III**

### **Syllabus:**

### **Unit – IV**

Linear Differential equation of  $n^{\text{th}}$  order with constant coefficient. Particular Integrals when RHS is of the form  $e^{ax}$ ,  $\sin ax$ ,  $x^n$ ,  $e^{ax} V$  and  $xV$  is function of  $x$

-10HRS

## Linear Equation with constant co-efficients.

### I. Linear Equ's with constant coefficients.

**Definition:-** A linear differential equation with constant coefficients is that in which the dependent variable and its differential coefficients occur only in the first degree and are not multiplied together, and the coefficients are all constant.

Thus, Equation.

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^y}{dx^{n-2}} + \dots + a_n y = x \quad (1)$$

where  $a_0, a_1, a_2, \dots, a_n$  are constants and  $x$  is any function of  $x$  is a linear differential equation of  $n$ th order.

### II The operator D.

The part  $\frac{dy}{dx}$  of the symbol  $\frac{dy}{dx}$  may be regarded as an operator, such that when it operates on  $y$ , the result is the derivative of  $y$ .

The operator denoted by  $D$ , which stands

$$\frac{d}{dx} = D, \quad \frac{d^2}{dx^2} = D^2, \quad \frac{d^3}{dx^3} = D^3, \quad \frac{d^4}{dx^4} = D^4, \quad \dots, \quad \frac{d^n}{dx^n} = D^n.$$

The equ<sup>n</sup>(1) can be written as

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^y}{dx^{n-2}} + \dots + a_n y = x.$$

This written in symbolic form becomes

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \dots + a_n y = x$$

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n) y = x$$

III.  $a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \dots + a_n = f(D)$  is regarded as a single operator, operating on  $y$ .

NOTE:- If  $y=y_1, y=y_2, \dots, y=y_n$  are  $n$  linearly independent solutions of then  $y=c_1y_1 + c_2y_2 + \dots + c_ny_n$  is general or complete solution of (i) where  $c_1, c_2, \dots, c_n$  are 'n' arbitrary constant.

#### IV Auxiliary Equations (A.E)

Consider the Differential Equation

$$(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \dots + a_n)y = 0 \quad (i)$$

$$\text{or } a_0D^n y + a_1D^{n-1}y + a_2D^{n-2}y + \dots + a_n y = 0 \quad (ii)$$

Let  $y = e^{mx}$  be a solution of (i) Then

$$Dy = me^{mx}, D^2y = m^2e^{mx}, D^3y = m^3e^{mx}, \dots, D^ny = m^ne^{mx}$$

Substituting the value of  $y, Dy, D^2y, \dots, D^ny$  in (ii)

$$a_0m^n e^{mx} + a_1m^{n-1}e^{mx} + \dots + a_n e^{mx} = 0$$

$$(a_0m^n + a_1m^{n-1} + \dots + a_n)e^{mx} = 0$$

Cancelling  $e^{mx}$  ( $\because e^{mx} \neq 0 \forall \text{any } m$ )

$$a_0m^n + a_1m^{n-1} + a_2m^{n-2} + \dots + a_n = 0 \quad (iii)$$

Thus  $e^{mx}$  will be a solution of (i) if  $m$  satisfy (iii) if  $m$  is a root of (iii)

Equ (iii) is called a auxiliary equation for the differential Equ (i)

Replacing  $m$  by  $D$  in (iii) we get

$$a_0D^n + a_1D^{n-1} + \dots + a_n = 0 \quad (iv)$$

This equ (iv) is called Auxiliary Equation

Definition of A.E :- The equation obtained by equating to zero the symbolic co-efficient of  $y$  is called Auxillary equation, provided  $D$  is taken as an algebraic quantity.

( $D$  is considered as an Algebraic quantity)

If the differential equation is  $f(D)y=0$

its auxilliary equation is  $f(D)=0$ .

NOTE: Replace  $D$  by  $\frac{dy}{dx}$  by 'm' to get the auxillary equation.

OR we can write Auxillary equation in Simplest form.

Consider the differential equation

$D^n + a_0 D^{n-1} + a_1 D^{n-2} + \dots + a_n y = x \quad \text{(1)}$   
if  $n^{\text{th}}$  order where  $a_0, a_1, a_2, a_3, \dots, a_n$  are all constant c. d. s.

Suppose  $y = e^{mx}$  be solution

Diff w.r.t  $x$   $Dy = me^{mx}$

Again diff w.r.t  $x$   $D^2y = m^2 e^{mx}$

$$D^ny = m^n e^{mx}$$

Substitute in (1)

$$(a_0 m^n + a_1 m^{n-1} + \dots + a_{n-1} m + a_n) e^{mx} = x$$

If  $x=0$  Then

$$(a_0 m^n + a_1 m^{n-1} + \dots + a_{n-1} m + a_n) e^{mx} = 0$$

$$\therefore e^{mx} \neq 0$$

$$a_0 m^n + a_1 m^{n-1} + \dots + a_{n-1} m + a_n = 0$$

This is called Auxillary equation.

To find the complementary function (C.F)

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n) y = 0 \quad (\text{A})$$

$$\text{The A.E for (A) is } a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n = 0 \quad (\text{ii})$$

Case I :- When all the roots of the A.E (ii)

are real and different.

Let  $m_1, m_2, m_3, \dots, m_n$  be the  $n$  real and different roots of (ii)

Then  $y = e^{m_1 x}, y = e^{m_2 x}, y = e^{m_3 x}, \dots, y = e^{m_n x}$

are 'n' independent solution of (A)

Hence the Complementary function or

Complete solution of (A) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

Case II :- When two roots of the A.E (ii) are equal and all others different.

When two roots of the auxiliary eqns

are same (when the roots are repeated)  
Then the 1<sup>st</sup> two factors of equation (A) are

$$(D - m_1)^2 y = 0 \quad (2)$$

$$(D - m_1)(D - m_1)y = 0$$

put  $(D - m_1)y = V$  it becomes

$$(D - m_1)V = 0 \text{ or } \frac{dV}{dx} = m_1 V$$

Separating variables  $\frac{dV}{V} = m_1 dx$

Integrating  $\log V = m_1 x + \log C$

$$\log V - \log C = m_1 x$$

$$\text{or } \log \frac{V}{C} = m_1 x$$

$$\log \frac{V}{C} = m_1 x \quad (3)$$

$$\frac{V}{C} = e^{m_1 x} \quad \therefore V = C e^{m_1 x}$$

$$(D - m_1) y = C e^{m_1 x} \quad (\because V = (D - m_1) y)$$

$$\frac{dy}{dx} - m_1 y = C e^{m_1 x}$$

which is a linear equation of the first order

$$I.F = e^{\int -m_1 dx} = e^{-m_1 x}$$

$$\text{It's solution is } y \cdot e^{-m_1 x} = \int C e^{m_1 x} e^{-m_1 x} dx + C_1$$

$$= \int C dx + C_1$$

$$= Cx + C_1$$

$$\therefore y = C_1 + Cx e^{m_1 x}$$

$$\text{or } y = C_1 + C_2 x e^{m_1 x}$$

Hence the complete Solution of (A)

$$\therefore y = C_1 + C_2 x e^{m_1 x} + C_3 e^{m_2 x} + \dots + C_n e^{m_n x}$$

$$\begin{aligned}
 &= e^{ax} (c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3 x} + \dots + c_n e^{m_n x} \\
 &= e^{ax} [c_1 (\cos \beta x + i \sin \beta x) + c_2 (\cos \beta x - i \sin \beta x)] + c_3 e^{m_3 x} + \dots \\
 &\quad \dots + c_n e^{m_n x} \quad (e^{i\theta} = \cos \theta + i \sin \theta) \\
 &= e^{ax} [(c_1 + c_2) \cos \beta x + i(c_1 - c_2) \sin \beta x] + c_3 e^{m_3 x} + \dots \\
 &= e^{ax} [c_3 \cos \beta x + c_4 \sin \beta x] + c_3 e^{m_3 x} + \dots + c_n e^{m_n x} \\
 &\quad (\text{Taking } c_1 + c_2 = c_3, c_1 - c_2 = c_4)
 \end{aligned}$$

NOTE:- After suitable adjusting constants  
 $e^{ax} (c_3 \cos \beta x + c_4 \sin \beta x)$  may also be  
written as  $e^{ax} c_3 \cos(\beta x + C_4)$  or  
 $e^{ax} c_3 \sin(\beta x + C_4)$

i.e. In case A.E has two equal pairs of  
imaginary roots i.e.  $a \pm i\beta, \pm i\beta$  then the  
complete solution is  
 $y = e^{ax} [(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x]$   
 $+ c_5 e^{m_5 x} + \dots + c_n e^{m_n x}$ .

### Working Rule

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \dots + a_n y = 0$$

(1) Write the equation in the symbol form  
 $(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n) y = 0$

(2) Write the auxiliary equation (A.E)

$$a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n = 0$$

Solve for D as if D were the ordinary  
algebraic quantity.

(3) From the roots of A.E write down the  
corresponding part of the complete solution as  
follows.  $\Rightarrow$

(4)

Roots of A.E	Corresponding part of C.S <small>(complete soln)</small>
1. One real root $m_1$	$c_1 e^{m_1 x}$
2. Two real and different roots $m_1, m_2$	$c_1 e^{m_1 x} + c_2 e^{m_2 x}$
3. Two real and equal roots $m_1, m_1$	$(c_1 + c_2 x) e^{m_1 x}$
4. Three real and equal roots $m_1, m_1, m_1$	$(c_1 + c_2 x + c_3 x^2) e^{m_1 x}$
5. One pair of complex roots $\alpha \pm i\beta$	$e^{\alpha x} [c_1 \cos \beta x + c_2 \sin \beta x]$
6. Two pairs of complex and equal roots $\alpha \pm i\beta, \alpha \pm i\beta$	$e^{\alpha x} [(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x]$
7. One pair of surd roots $\alpha \pm \sqrt{\beta}$	$e^{\alpha x} [c_1 \cosh x \sqrt{\beta} + c_2 \sinh x \sqrt{\beta}]$
8. Two pair of surd & equal roots $\alpha \pm \sqrt{\beta}, \alpha \pm \sqrt{\beta}$	$e^{\alpha x} [(c_1 + c_2 x) \cosh x \sqrt{\beta} + (c_3 + c_4 x) \sinh x \sqrt{\beta}]$

Example:-

$$(1) \text{ Solve } \frac{d^3y}{dx^3} - 13\frac{dy}{dx} + 12y = 0$$

Solu<sup>n</sup>:- The given equation is

$$\frac{d^3y}{dx^3} - 13\frac{dy}{dx} + 12y = 0$$

In the form of symbol D, the equation becomes

$$(D^3 - 13D + 12)y = 0$$

The auxiliary equation is

$$m^3 - 13m + 12 = 0$$

$$(m-1)(m^2 + m + 12) = 0$$

$$(m-1)(m^2 + m + 12) = 0$$

$$\Rightarrow (m-1)(m+4)(m-3) = 0$$

$\therefore$  The roots are 1, -4, 3 which are distinct

Hence the solution is

$$y = C_1 e^x + C_2 e^{-4x} + C_3 e^{3x}$$

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -13 & +12 \\ & & -1 & 1 & -12 \\ \hline & 1 & -1 & -12 & 0 \end{array}$$

$$\therefore (D^2 - D - 12) = 0$$

$$m^2 - m - 12 = 0$$

$$m^2 + 4m - 3m - 12 = 0$$

$$m(m+4) - 3(m+4)$$

$$(m+4)(m-3) = 0$$

$$(2) \frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 0$$

Given equ<sup>n</sup> in the symbolic form

$$(D^2 - 3D - 4)y = 0$$

A.B is

$$\begin{aligned} D^2 - 3D - 4 &= 0 \\ (D-4)(D+1) &= 0 \end{aligned}$$

$$D = 4, -1$$

$\therefore$  The complete solu<sup>n</sup> is  $y = C_1 e^{4x} + C_2 e^{-x}$ ,

$$(3) \text{ Solve } \frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = 0$$

Given equ<sup>n</sup> in symbolic form  $(D^3 - 7D - 6) = 0$

$$\begin{array}{r|rrr} -1 & 1 & 0 & -7 & -6 \\ & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & 0 \end{array}$$

$$A.B: D^3 - 7D - 6 = 0$$

Put  $D = -1$  L.H.S of A.B =  $-1 + 7 - 6 = 0$

$(D+1)$  is a factor.

$$\therefore A.B: (D+1)(D^2 - D - 6) = 0$$

Qe.

$\therefore$  The complete solution is  $y = C_1 e^x + C_2 e^{2x} + C_3 e^{3x}$ . (5)

(4) <sup>2014, 2019</sup> Solve  $(D^2 - 4D + 1)y = 0$

$\therefore$  A.R is  $D^2 - 4D + 1 = 0$   
 $m^2 - 4m + 1 = 0$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$m = \frac{-(-4) \pm \sqrt{(-4)^2 - 4 \times 1 \times 1}}{2 \times 1}, m = \frac{4 \pm \sqrt{16 - 4}}{2}$$

$$= \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2}$$

$$\therefore m = 2 \pm \sqrt{3}$$

$$m_1 = 2 + \sqrt{3}, m_2 = 2 - \sqrt{3}$$

Complete solu<sup>n</sup> is  $y = C_1 e^{(2+\sqrt{3})x} + C_2 e^{(2-\sqrt{3})x}$

<sup>2016</sup> (5) Solve  $(D^2 - 4D + 4)y = 0$

Solu<sup>n</sup>: A.R is  $m^2 - 4m + 4 = 0$

$$m^2 - 2m - 2m + 4 = 0$$

$$m(m-2) - 2(m-2) = 0$$

$$(m-2)(m-2)$$

$$m = 2, 2$$

$\therefore$  The complete solu<sup>n</sup> is

$$y = (C_1 + C_2 x)e^{2x}$$

7) <sup>2016</sup>  
 Solve  $D^4y + 8D^2y + 16y = 0$   
 $\Rightarrow C(D^4 + 8D^2 + 16)y = 0$   
 A.B is  $D^4 + 8D^2 + 16 = 0$   
 $m^4 + 8m^2 + 16 = 0$   
 $(m^2 + 4)^2 = 0$   
 $(m^2 + 4)(m^2 + 4) = 0$   
 $m^2 + 4 = 0, \quad m^2 + 4 = 0$   
 $m^2 = -4 \quad m^2 = -4$   
 $m = \pm \sqrt{-4}$   
 $m = \pm 2i$   
 The complete soln is  $y = e^{dx} [C_1 \cos 2x + C_2 \sin 2x] + C_3 e^{dx} \cos 2x + C_4 e^{dx} \sin 2x$   
 $y = (C_1 + C_2 x) \cos 2x + (C_3 + C_4 x) \sin 2x$

8) Solve  $(D^4 - D^3 - 9D^2 - 11D - 4)y = 0$   
 The given eqn is  $(D^4 - D^3 - 9D^2 - 11D - 4)y = 0$   
 The A.B is  $m^4 - m^3 - 9m^2 - 11m - 4 = 0$   
 $\therefore m=1$  is root thus  
 $(m+1)$  is factor  
 $\therefore (m+1)(m^3 - 2m^2 - 7m - 4) = 0$   
 (By actual division)  
 Again by inspection  $-1$ , is a root  
 $\therefore (m+1)^2(m^2 - 3m - 4) = 0$   
 $(m+1)^2(m+1)(m-4) = 0$   
 $(m+1)^3(m-4) = 0$   
 The A.B are  $-1, -1, -1, 4$   
 $\therefore$  Hence the complete soln is  
 $y = C_1 e^{-4x} + C_2 + C_3 x + C_4 x^2 e^{-x}$

$$\begin{array}{r|rrrrr} & 1 & 1 & -1 & -9 & -11 & -4 \\ & & -1 & 2 & -9 & -4 \\ \hline & 1 & -2 & -9 & -4 & 0 \\ & & 1 & -1 & -8 & 0 \\ \hline & & & 1 & -8 & 0 \end{array}$$

$$\begin{array}{r|rrrr} +1 & -11 & 0 & 9 & 3 & -1 \\ & \diagup & \diagup & \diagup & \diagup & \\ & 1 & 1 & -8 & 0 \\ \hline -1 & 1 & -2 & -7 & -4 & 0 \\ & & & 1 & -3 & 0 \end{array}$$

$$(9) \text{ Solve } \frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 8y = 0 \quad (6)$$

Solu<sup>n</sup>: - The given equ<sup>n</sup> in the form  
 $(D^3 - 2D^2 + 4D - 8)y = 0$

The A. R is

$$m^3 - 2m^2 + 4m - 8 = 0$$

$\therefore$  put  $m=2$  is a root

$$(m-2)(m^2+4)=0$$

$\therefore$  The roots are  $2, \pm 2i$

$\therefore$  The complete solu<sup>n</sup> is

$$y = c_1 e^{2x} + c_2 \cos 2x + c_3 \sin 2x$$

$$(10) \text{ Solve } (D^3 - D^2 - D - 2)y = 0$$

Solu<sup>n</sup>: - The given equ<sup>n</sup> is

$$(D^3 - D^2 - D - 2)y = 0$$

The A. R is  $m^3 - m^2 - m - 2 = 0$

$$(m-2)(m^2 + m + 1) = 0$$

$$m = 2, \quad m^2 + m + 1 = 0$$

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\therefore m = -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

Hence Required solu<sup>n</sup> is

$$y = c_1 e^{2x} + e^{-\frac{1}{2}x} \left( c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x \right)$$

(12) <sup>2019</sup>  
Solve  $(D^2 + 25)y = 0$

A.R is  $D^2 + 25 = 0 \Rightarrow m^2 + 25 = 0$

$$m^2 = -25$$

$$m = \pm\sqrt{25} = \pm\sqrt{-25}$$

$$m = \pm 5i$$

The complete soln is

$$\begin{aligned}y &= e^{0x} [C_1 \cos 5x + C_2 \sin 5x] \\&= C_1 + C_2 x \cos 5x + (C_3 + C_4 x) \sin 5x\end{aligned}$$

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(13) <sup>2018</sup>  
Solve  $D^2 - 5D + 4 = 0$

The A.R is  $m^2 - 5m + 4 = 0$

$$m^2 - 4m - m + 4 = 0$$

$$m(m-4) - 1(m-4) = 0$$

$$m = 4, 1$$

The O.P is

$$y = \phi_1(yt + u) + \phi_2(y + u)$$

## (7)

### General Method of Finding Particular Integral.

Solve  $(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n) Y = X - (1)$   
 Consider  $(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \dots + a_n) Y = 0 - (2)$

$$Y = c_1 y_1 + c_2 y_2 + c_3 y_3 + \dots + c_n y_n$$

If  $y = u$  is particular solution of (1)  
 Then the general solution of (1) is given by

$$G.S = Y + u,$$

$y$  is called complementary function of (1)  
 $u$  is " particular solution of (1)

$$\therefore Y = C.F + P.I$$

NOTE: The inverse operator  $\frac{1}{f(D)}$   
 Definition:-  $\frac{1}{f(D)} X$  is a function of  $x$ , free from  
 arbitrary constant, which when operated  
 upon by  $f(D)$  give  $X$ .

$$f(D). \frac{1}{f(D)} X = X$$

Thus  $f(D)$  &  $\frac{1}{f(D)}$  are inverse operators.  
 $\therefore f(D)$  play the role of differential operator,  
 $\frac{1}{f(D)}$  play the role of integral operator.

Thm: (1) P.T  $\frac{1}{f(D)} x$  is the particular integral of the equation  $f(D)y = x$

Pf: The equ<sup>n</sup> is  $f(D)y = x \quad \text{--- (1)}$   
 put  $y = \frac{1}{f(D)} x$ ,  $x = x = \text{RHS of (1)}$   
 LHS of (1) =  $f(D) \cancel{\frac{1}{f(D)} x}$   
 $\therefore y = \frac{1}{f(D)} x$  is soln of (1)  
 $\therefore$  It contains no arbitrary constant.  
 $\therefore$  It is particular integral.

i. It is particular integral, no arbitrary constant being added.

Thm: (2) - P.T  $\frac{1}{D-\alpha} x = e^{\alpha x} \int x e^{-\alpha x} dx$ , no arbitrary

constant being added.

Pf: Let  $\frac{1}{D-\alpha} x = y$   
 operating on both sides by  $(D-\alpha)$ , we get

$$(D-\alpha) \cdot \frac{1}{(D-\alpha)} x = (D-\alpha) y$$

$$x = D y - \cancel{dy} = \cancel{\frac{dy}{dx}} - \alpha y \quad \text{or}$$

$$\cancel{\frac{dy}{dx}} - \alpha y = x.$$

which is linear in  $y$ .

$$I.F = e^{\int \alpha dx} = e^{-\alpha x}$$

$\therefore$  solution is  $y \cdot e^{-\alpha x} = \int x e^{-\alpha x} dx$ , no arbitrary constant added

$(\because y = \frac{1}{D-\alpha} x) \text{ containing no arbitrary constant}$

$$\boxed{y = e^{\alpha x} \int x e^{-\alpha x} dx}$$

$$\boxed{OR \quad \frac{1}{(D-\alpha)} x = e^{\alpha x} \int x e^{-\alpha x} dv}$$

## Standard Case of Particular Integrals (8)

Case I:-  $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$  if  $f(a) \neq 0$

pf:- By successive Differentiation.

$$D e^{ax} = a e^{ax} \text{ diff w.r.t } x$$

$$D^2 e^{ax} = a^2 e^{ax}$$

$$D^n e^{ax} = a^n e^{ax}$$

$$\therefore f(D) e^{ax} = f(a) e^{ax}$$

operating both sides by  $\frac{1}{f(D)}$

$$\frac{1}{f(D)} \cdot f(D) e^{ax} = \frac{1}{f(D)} f(a) e^{ax}$$

$$\therefore e^{ax} = \frac{1}{f(D)} f(a) e^{ax} \quad \underline{\text{OR}}$$

$$e^{ax} = f(a) \frac{1}{f(D)} e^{ax} \quad (\because f(a) \text{ is constant})$$

Dividing both side by  $f(a)$  which is not zero

$$\frac{1}{f(a)} e^{ax} = \frac{1}{f(D)} e^{ax}$$

Hence

$$\boxed{\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}}$$

**NOTE:-** To evaluate  $\frac{1}{f(D)} e^{ax}$ , put  $D=0$ , ( $f(a) \neq 0$ )

## Standard Cases of Particular Integrals (8)

**Case I:-**  $\frac{1}{f(D)} e^{ax}$ ,  $\frac{1}{f(a)} e^{ax}$  if  $f(a) \neq 0$

**Pf:-** By successive Differentiation.

$$D e^{ax} = a e^{ax} \text{ diff w.r.t } x$$

$$D^2 e^{ax} = a^2 e^{ax}$$

$$D^n e^{ax} = a^n e^{ax}$$

$$\therefore f(D) e^{ax} = f(a) e^{ax}$$

Operating both sides by  $\frac{1}{f(D)}$

$$\frac{1}{f(D)} \cdot f(D) e^{ax} = \frac{1}{f(D)} f(a) e^{ax}$$

$$\therefore e^{ax} = \frac{1}{f(D)} f(a) e^{ax} \quad \underline{\text{OR}}$$

$$e^{ax} = f(a) \frac{e^{ax}}{f(D)} \quad (\because f(a) \text{ is constant})$$

Dividing both side by  $f(a)$  which is not zero

$$\frac{1}{f(a)} e^{ax} = \frac{1}{f(D)} e^{ax}$$

Hence

$$\boxed{\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}}$$

**NOTE:-** To evaluate  $\frac{1}{f(D)} e^{ax}$ , put  $D=a$ , ( $f(a) \neq 0$ )

Exemplu: S.T  $\frac{1}{(D-a)^n} e^{ax} = \frac{x^n}{n!} e^{ax}$ . (8)

$$\begin{aligned}
 \text{Solu: } & \frac{1}{(D-a)^n} e^{ax} = \frac{1}{(D-a)^{n-1}} \cdot \frac{1}{(D-a)} e^{ax} \\
 & = \frac{1}{(D-a)^{n-1}} e^{ax} \int e^{-ax} \cdot e^{ax} dx \\
 & = \frac{1}{(D-a)^{n-1}} e^{ax} \int dx \\
 & = \frac{1}{(D-a)^{n-1}} \frac{x}{1!} e^{ax}. \quad (1) \\
 & = \frac{1}{(D-a)^{n-2}} \frac{1}{(D-a)} x e^{ax} = \frac{1}{(D-a)^{n-2}} e^{ax} \int e^{-ax} x dx \\
 & = \frac{1}{(D-a)^{n-2}} \cdot e^{ax} \int x dx = \frac{1}{(D-a)^{n-2}} \cdot \frac{x^2}{2!} e^{ax} \quad (2) \\
 & = \frac{1}{(D-a)^{n-3}} \cdot \frac{1}{(D-a)} \frac{x^2}{2!} e^{ax} \\
 & = \frac{1}{(D-a)^{n-3}} e^{ax} \int e^{-ax} \cdot \frac{x^2}{2!} e^{ax} dx \\
 & = \frac{1}{(D-a)^{n-3}} e^{ax} \frac{x^3}{3!} \quad (3) \\
 & \therefore \text{Generalising from (i) (ii) & (iii)}
 \end{aligned}$$

$$\begin{aligned}
 \frac{1}{(D-a)^n} e^{ax} &= \frac{1}{(D-a)^{n-n}} \frac{x^n}{n!} e^{ax} \\
 \boxed{\frac{1}{(D-a)^n} e^{ax} &= \frac{x^n e^{ax}}{n!}}
 \end{aligned}$$

(9)

Example:-

$$(1) \text{ Solve } \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = e^{-x}.$$

Solu<sup>n</sup>: The given equn is  $(D^2 + D + 1)y = e^{-x}$

$$\text{A.E is } D^2 + D + 1 = 0$$

$$D = \frac{-1 \pm \sqrt{1-4}}{2} = -1 \pm i\sqrt{3} = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\text{C.F} = e^{\frac{-x}{2}} \left[ C_1 \cos \frac{\sqrt{3}}{2}x + C_2 \sin \frac{\sqrt{3}}{2}x \right]$$

$$\text{P.I} = \frac{1}{(D^2 + D + 1)} e^{-x} \quad |_{f(D)}$$

$$= \frac{1}{(-1)^2 + (-1) + 1} e^{-x}$$

$$= \frac{1}{1} e^{-x} = e^{-x}$$

$$\text{put } D = -1$$

$$(\because \frac{1}{f(D)} e^{ax} \quad a = -1, \text{ put } D = -1)$$

$\therefore$  The complete solu<sup>n</sup> is  $y = \text{C.F} + \text{P.I}$

$$y = e^{\frac{-x}{2}} \left[ C_1 \cos \frac{\sqrt{3}}{2}x + C_2 \sin \frac{\sqrt{3}}{2}x \right] + e^{-x}$$

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$$(2) \text{ Solve } (D^2 + 1)y = 8 + 5e^x$$

$$(3) \text{ Solve } (D^2 - 1)y = 3 - 5e^{2x}$$

$$(4) \text{ Solve } (D^2 - 6D + 9)y = 4e^{3x}$$

$$(5) \text{ Solve } (D^2 - 5D + 6)y = x.$$

$$(6) \text{ Solve } (D^3 - 3D^2 + 4D - 2)y = e^x$$

$$(7) \text{ Solve } (D^2 + 1)y = 8 + 5e^x \quad D^2 = 1 \quad D = \pm i \quad D = \pm \sqrt{1} = \pm i$$

$$\text{A.E is } D^2 + 1 = 0, \quad D = \pm i$$

$$\text{C.F} = e^{0x} \left[ C_1 \cos x + C_2 \sin x \right]$$

$$\text{P.F} = \frac{1}{(D^2 + 1)} 8 + 5e^x = \frac{1}{(D^2 + 1)} (8e^{0x} + 5e^x) \\ = 8 \cdot \frac{1}{D^2 + 1} e^{0x} + 5 \cdot \frac{1}{D^2 + 1} e^x$$

$$(3) \text{ Solve } (D^2 - 1)y = 3 - 5e^{2x}$$

$$= 8 \cdot \frac{1}{D+1} e^{0x} + 5 \cdot \frac{1}{(2)^2 - 1} e^{2x}$$

$$= -8 + 5 e^{2x}$$

$\therefore$  Complete soln is  $y = C.F + P.I$

$$y = e^{0x} [C_1 \cos x + C_2 \sin x] - 8 + \frac{5}{3} e^{2x}$$

$$(3) \text{ Solve } (D^2 - 1)y = 3 - 5e^{2x}$$

$$\text{Solu}^n. \quad A.E \text{ is } D^2 - 1 = 0$$

$$D^2 = 1$$

$$D = \pm \sqrt{1} = 1$$

$$C.F = C_1 e^x + C_2 e^{-x}$$

$$P.I = \frac{1}{(D^2 - 1)} (3 - 5e^{2x}) =$$

$$= \frac{1}{(D^2 - 1)} (3e^{0x} + 5e^{2x}) = -3 + \frac{5}{3} e^{2x}$$

$$= \frac{3}{(0)^2 - 1} + 5 \cdot \frac{1}{(2)^2 - 1} e^{2x}$$

$\therefore$  The complete soln is

$$y = C.F + P.I$$

$$= C_1 e^x + C_2 e^{-x} - 3 + \frac{5}{3} e^{2x}$$

$$(4) \text{ Solve } (D^2 - 6D + 9)y = 4e^{3x}$$

Solve particular integral.

$$\text{p.f. particular integral } y = \frac{4e^{3x}}{D^2 - 6D + 9}$$

$$= \frac{4e^{3x}}{(3)^2 + 6 \times 3 + 9} = \frac{4e^{3x}}{9 + 18 + 9}$$

$$\text{Again diff } f = \frac{4e^{3x}}{36} = \frac{2e^{3x}}{18} \text{ as } \int \frac{1}{x^2} dx = \frac{1}{x} \text{ so } \frac{1}{36} = \frac{1}{36}$$

(5) Solve  $\frac{d^3y}{dx^3} + y = 3 + 5e^x$ .

Given equation in symbolic form  $(D^3+1)y = 3 + 5e^x$

A.E  $D^3 + 1 = 0 \Rightarrow D = -1$

$$(D+1)(D^2 - D + 1) = 0$$

$$D = -1, \quad 1 \pm \sqrt{-4} \Rightarrow D = -1, \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$C.F = C_1 e^{-x} + e^{\frac{x}{2}} \left[ C_2 \cos \frac{\sqrt{3}}{2}x + C_3 \sin \frac{\sqrt{3}}{2}x \right]$$

$$P.I = \frac{1}{(D^3+1)} (3 + 5e^x) = \frac{1}{(D^3+1)} (3e^{0x} + 5e^x)$$

$$= 3 \cdot \frac{1}{(D^3+1)} e^{0x} + 5 \cdot \frac{1}{(D^3+1)} e^x$$

$$= 3 \cdot \frac{1}{(0+1)} e^{0x} + 5 \cdot \frac{1}{(1)^3+1} e^x = 3 + \frac{5}{2} e^x$$

The complete soln is  $y = C.F + P.I$

$$y = C_1 e^{-x} + e^{\frac{x}{2}} \left[ C_2 \cos \frac{\sqrt{3}}{2}x + C_3 \sin \frac{\sqrt{3}}{2}x \right] + 3 + \frac{5}{2} e^x$$

Case II:- To Find Particular Integral when  
X is of the form  $x = \sin ax$ .

OR P.T  $\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$ .

Pf:- By successive differentiation, we have

$$D \sin ax = a \cos ax$$

$$D^2 \sin ax = -a^2 \sin ax$$

$$(D^2)^1 \sin ax = (-a^2)^1 \sin ax$$

$$D^3 \sin ax = -a^3 \cos ax$$

$$D^4 \sin ax = a^4 \cos ax$$

$$(D^2)^2 \sin ax = (-a^2)^2 \sin ax$$

$$\underline{\underline{(D^2)^n \sin ax}} = \underline{\underline{(-a^2)^n \sin ax}}$$

$$\therefore f(D^2) \sin ax = f(-a^2) \sin ax$$

Operating upon both sides by  $\frac{1}{f(D^2)}$ , we have

$$\Rightarrow \frac{1}{f(D^2)} f(D^2) \sin ax = \frac{1}{f(D^2)} f(-a^2) \sin ax$$

$$\sin ax = f(-a^2) \frac{1}{f(D^2)} \sin ax$$

Dividing both sides  $f(-a^2)$  which is not zero

$$\frac{1}{f(-a^2)} \sin ax = \frac{1}{f(D^2)} \sin ax$$

Hence 
$$\boxed{\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax} \quad \text{where } f(-a^2) \neq 0$$

$$\text{case III: } \frac{D^4}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax. \quad (11)$$

$\Rightarrow$  By successive differentiation.

$$D(\cos ax) = -a \sin ax$$

$$D^2(\cos ax) = -a^2 \cos ax = (-a^2)' \cos ax$$

$$D^3(\cos ax) = D^3 \sin ax$$

$$D^4(\cos ax) = a^4 \cos ax = -(-a^2)^2 \cos ax$$

$$D^n(\cos ax) = (-a^2)^n \cos ax$$

$$f(D^2) \cos ax = f(-a^2) \cos ax$$

operating by  $\frac{1}{f(D^2)}$

$$\frac{1}{f(D^2)} [f(D^2) \cos ax] = \frac{1}{f(D^2)} f(-a^2) \cos ax$$

$$\cos ax = f(-a^2) \frac{1}{f(-a^2)} \cos ax$$

Dividing both sides  $f(-a^2)$  which is not zero

$$\frac{1}{f(-a^2)} \cos ax = \frac{1}{f(D^2)} \cos ax$$

## Important Formulas

$$= \frac{1}{(r-D)} = (r-D)^{-1}$$

$$(1) (r-D)^{-1} = 1 + D + D^2 + D^3 + \dots$$

$$(2) (r+D)^{-1} = 1 - D + D^2 - D^3 + \dots$$

$$(3) (r-D)^2 = 1 + 2D + 3D^2 + 4D^3 + \dots$$

$$(4) (r+D)^2 = 1 - 2D + 3D^2 - 4D^3 + \dots$$

$$(5) (r-D)^3 = 1 + 3D + 6D^2 + 9D^3 + \dots$$

$$(6) (r+D)^3 = 1 - 3D + 6D^2 - 9D^3 + \dots$$


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**Example:-** (1) Find particular integral of

$$(D^2 - 6D + 9)y = 4e^{3x}.$$

P.I.  $\frac{1}{(D-a)^n} F(D)^2 \underset{n}{\cancel{x^n}} \frac{e^{ax}}{(D-a)^n}$

$$y = \frac{4e^{3x}}{D^2 - 6D + 9} = 4e^{3x} \underset{2!}{\cancel{x^2}}$$

$$= \frac{4e^{3x} x^2}{2x!} = 2e^{3x} x^2$$

$$= 2x^2 e^{3x}.$$


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(2) Solve  $\frac{d^2y}{dx^2} + 2 \frac{dy}{dx} + y = 2e^{2x}.$

Soln:- The given differential equation can be written as  $(D^2 + 2D + 1)y = 2e^{2x}$

The A.E is  $m^2 + 2m + 1 = 0$   
 $(m+1)^2 = 0$

i.e. The roots are  $-1, -1$

$\therefore$  C.F is  $C_1 e^{-x} + C_2 x e^{-x}$

$$\text{P.I} = \frac{1}{(D^2 + 2D + 1)} 2e^{2x}$$

$$= \frac{1}{(D+1)^2} 2e^{2x} = \frac{2}{9} e^{2x}$$

Hence the solution is  
 $y = (c_1 + c_2 x) e^x + \frac{2}{9} e^{2x}$

(3) Solve  $\frac{d^3y}{dx^3} - 3\frac{dy}{dx^2} + 4\frac{dy}{dx} - 2y = e^x$ .

Solu<sup>n</sup>: The given D. B can be written as  
 $(D^3 - 3D^2 + 4D - 2)y = e^x$ .

A.E  $m^3 - 3m^2 + 4m - 2 = 0$

$\therefore m=1$  is a root

$$(m-1)(m^2 - 2m + 2) = 0$$

$$m=1 \text{ or } m=1 \pm i$$

$$\therefore C.F = c_1 e^x + e^x (c_2 \cos x + c_3 \sin x)$$

$$P.I = \frac{1}{(D^3 - 3D^2 + 4D - 2)}$$

$\therefore F(m)=0$ , i.e. 1 is root of equ<sup>n</sup>  $F(1)=0$

$$P.I = \frac{1}{(D-1)(D^2 - 2D + 2)}$$

$$= \frac{1}{1^2 - 2 \cdot 1 + 2} \cdot \frac{x}{1!} e^x = x e^x$$

Hence solu<sup>n</sup> is  $y = c_1 e^x + e^x (c_2 \cos x + c_3 \sin x) + x e^x$

$$y = e^x [c_1 + c_2 \cos x + c_3 \sin x + x]$$

(4) Solve  $(D^3 + 1)y = 3 + e^{-x} + 5e^{2x}$

Solu<sup>n</sup>: The given D. B can be written as

$$(D^3 + 1)y = 3 + e^{-x} + 5e^{2x}$$

The A.E is  $m^3 + 1 = 0$

$$(m+1)(m^2 - m + 1) = 0$$

$$\therefore m = -1, \text{ or } m = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

Hence the solution is  
 $y = (C_1 + C_2 x) e^{-x} + \frac{2}{9} e^{2x}$

③ Solve  $\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^x$ .

Solu<sup>n</sup>: The given D. B can be written as  
 $(D^3 - 3D^2 + 4D - 2)y = e^x$ .

A.E  $m^3 - 3m^2 + 4m - 2 = 0$

$\therefore m=1$  is a root

$$(m-1)(m^2 - 2m + 2) = 0$$

$$m=1 \text{ or } m=1 \pm i$$

$$\therefore C_F = C_1 e^x + e^x (C_2 \cos x + C_3 \sin x)$$

$$P.I = \frac{1}{(D^3 - 3D^2 + 4D - 2)}$$

$\therefore F(m)=0$ , i.e. 1 is root of equ<sup>n</sup>  $F(D)=0$

$$P.I = \frac{1}{(D-1)(D^2 - 2D + 2)}$$

$$= \frac{1}{1^2 - 2 \cdot 1 + 2} \cdot \frac{x}{1!} e^x = x e^x$$

Hence solu<sup>n</sup> is  $y = C_1 e^x + e^x (C_2 \cos x + C_3 \sin x) + x e^x$

$$y = e^x [C_1 + C_2 \cos x + C_3 \sin x + x]$$

④ solve  $(D^3 + 1)y = 3 + e^{-x} + 5e^{2x}$

Solu<sup>n</sup>: The given D. B can be written as  
 $(D^3 + 1)y = 3 + e^{-x} + 5e^{2x}$

The A.E is  $m^3 + 1 = 0$

$$(m+1)(m^2 - m + 1) = 0$$

$$\therefore m = -1, \text{ or } m = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$\text{Thus } C.F = c_1 e^x + e^{2x} \left[ c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x \right]$$

$$\begin{aligned} P.I &= \frac{1}{(D^3+1)} (3 + e^x + 5e^{2x}) \\ &= \frac{1}{(D^3+1)} \cancel{x} 3 + \frac{1}{(D^3+1)} \cancel{e^x} + \cancel{\frac{5}{(D^3+1)} e^{2x}} \end{aligned}$$

Consider

$$\therefore = \frac{1}{(D^3+1)} 3 \times e^{0x} + \frac{1}{(D^3+1)(D^2-D+1)} e^x + \frac{5}{(D^3+1)} e^{2x}$$

$$= 3 \times \frac{1}{(0+1)} e^{0x} + \frac{1}{1+1+1} \cancel{x} \cancel{e^x} + \cancel{\frac{5}{8+1} e^{2x}}$$

$$= 3 + \cancel{\frac{5}{9} e^{2x}}$$

Hence solutions

$$y = c_1 e^x + e^{2x} \left[ c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x \right] + 3 + \cancel{\frac{5}{9} e^{2x}}$$

Q17 Solve  $\frac{d^2y}{dx^2} - 4 \frac{dy}{dx} + 2y = \sin 3x$

Solution  
The given equn in symbolic form

$$(D^2 - 4D + 2)y = \sin 3x$$

A.B is  $D^2 - 4D + 2 = 0$   
 $D^2 - 2D - 2D + 2$   
 $D(D-2)$

$$\text{Thus } C.F = c_1 e^{-x} + e^{\frac{x}{2}} \left[ c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x \right]$$

$$\begin{aligned} P.I &= \frac{1}{(D^3+1)} (3 + e^{-x} + 5e^{2x}) \\ &= \frac{1}{(D^3+1)} \cancel{3} + \frac{1}{(D^3+1)} \cancel{e^{-x}} + \cancel{\frac{5}{(D^3+1)} e^{2x}} \end{aligned}$$

Consider

$$\therefore = \frac{1}{(D^3+1)} 3x e^{0x} + \frac{1}{(D^3+1)(D^2-D+1)} e^{-x} + \frac{5}{(D^3+1)} e^{2x}$$

$$= 3x \frac{1}{(0+1)} e^{0x} + \frac{1}{1+1+1} \cancel{x} \cancel{e^{-x}} + \cancel{\frac{5}{8+1} e^{2x}}$$

$$= 3 + \cancel{\frac{3}{3} e^{-x}} + \cancel{\frac{5}{9} e^{2x}}$$

Hence solutions

$$y = c_1 e^{-x} + e^{\frac{x}{2}} \left[ c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x \right] + 3 + \cancel{\frac{3}{3} e^{-x}} + \cancel{\frac{5}{9} e^{2x}}$$

Q.5 Solve  $\frac{d^2y}{dx^2} - 4 \frac{dy}{dx} + 2y = \sin 3x$

Solution:- The given equn in symbolic form

$$(D^2 - 4D + 2)y = \sin 3x$$

A.D is  $D^2 - 4D + 2 = 0$   
 $D^2 - 2D - 2D + 2$   
 $D(D-2)$

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Q. P.

$$(6) \text{ Solve } (D^2 - 5D + 6)yz \sin 3x. \cos 2x \\ \Rightarrow D^2 - 5D + 6 = 0,$$

$$\text{A.E } m^2 - 5m + 6 = 0$$

$$m^2 - 2m - 3m + 6 = 0$$

$$(m-2)(m-3) = 0$$

$$m = 2, 3$$

$$CF = C_1 e^{2x} + C_3 e^{3x}$$

$$P.I = \frac{\sin 3x. \cos 2x}{D^2 - 5D + 6}$$

multiply & divide by 2

$$= \frac{1}{2} \frac{1}{D^2 - 5D + 6} [2 \sin 3x. \cos 2x]$$

$$\left. \begin{array}{l} 2 \sin A. \cos B = \sin(A+B) + \sin(A-B) \\ = \frac{1}{2} \frac{1}{(D^2 - 5D + 6)} \times [\sin 5x + \sin x] \end{array} \right\}$$

$$= \frac{1}{2} \frac{1}{(D^2 - 5D + 6)} \sin 5x + \frac{1}{2} \frac{1}{(D^2 - 5D + 6)} \sin x \\ (\because \frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax)$$

$$= \frac{1}{2} \frac{1}{(5^2 - 5D + 6)} \sin 5x + \frac{1}{2} \frac{1}{(-1)^2 - 5D + 6} \sin x$$

$$= \frac{1}{2} \left[ \frac{1}{25 - 5D + 6} \sin 5x + \frac{1}{2(5 - 5D)} \sin x \right]$$

$$= -\frac{1}{2} \left( \frac{1}{19 + 5D} \sin 5x \right) + \frac{1}{2} \left( \frac{1}{5 - 5D} \sin x \right)$$

$$= -\frac{1}{2} \frac{1}{19 + 5D} \times \frac{(9 - 5D)}{(9 + 5D)} \sin 5x + \frac{1}{2} \frac{(5 + 5D) \sin x}{(5)^2 - (5D)^2}$$

$$= -\frac{1}{2} \frac{(19 - 5D) \sin 5x}{361 - 25D^2} + \frac{1}{2} \times \frac{(5 + 5D) \sin x}{25 - 25D^2}$$

$$= -\frac{1}{2} \frac{(19 - 5D) \sin 5x}{361 - 25(5D^2)} + \frac{1}{2} \left( \frac{5 + 5D) \sin x}{25 - 25(-1)^2} \right)$$

$$\begin{aligned}
 &= \frac{1}{2} \frac{(19-50)\sin 5x}{3c_1 + 625} + \frac{1}{2} \frac{(5+50)\sin u}{25+25} \\
 &= \frac{1}{2} \frac{(19-50)\sin 5x}{986} + \frac{1}{2} \frac{(5+50)\sin u}{50} \\
 &= -\frac{1}{2} \frac{19 \sin 5x - 5 \times 5 \cos 5x}{986} + \frac{1}{2} \times \frac{5 \sin u + 5 \cos u}{50} \\
 &= -\left( \frac{9 \sin 5x + 25 \cos 5x}{1972} \right) + \frac{1}{2} \frac{5(\sin u + \cos u)}{50} \\
 &= -\frac{19 \sin 5x + 25 \cos 5x}{1972} + \frac{\sin u + \cos u}{20}
 \end{aligned}$$

Complete Soln is

$$y = C.F + P.I$$

$$y = C_1 e^{2x} + C_2 e^{3x} - \frac{19 \sin 5x + 25 \cos 5x}{1972} + \frac{\sin u + \cos u}{20}$$


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$$(7) . \text{ Solve } (D^2 + 9)y = \sin 2u$$

$$\Rightarrow A.B m^2 + 9 = 0$$

$$m = \pm 3i$$

$$C.F = C_1 \cos 3u + C_2 \sin 3u$$

$$P.I = \frac{\sin 2u}{D^2 + 9} = \frac{\sin 2u}{(-2)^2 + 9} = \frac{1}{5} \sin 2u$$

$$y = P.I + C.F$$

$$y = C_1 \cos 3u + C_2 \sin 3u + \frac{1}{5} \sin 2u$$


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2014 Solve  $(D^3 - D^2 - 6D)y = x^2 + 1$  (14)

$$D^3 - D^2 - 6D = 0$$

$$D(D^2 - D - 6) = 0$$

$$D = 0, D^2 - D - 6 = 0$$

$$D^2 - 3D + 2D - 6 = 0$$

$$D(D-3) + 2(D-3) = 0$$

$$(D-3)(D+2) = 0$$

$$D = -2, 3$$

$$CF = C_1 e^{0x} + C_2 e^{-2x} + C_3 e^{3x}$$

$$= C_1 + C_2 e^{-2x} + C_3 e^{3x}$$

$$P.I. = \frac{1}{D^3 - D^2 - 6D} x(x^2 + 1)$$

$$= -\frac{1}{6D} \left( -\frac{D^2}{6} + \frac{D}{6} + 1 \right)$$

$$= -\frac{1}{6D} \left[ 1 + \left( -\frac{D^2}{6} + \frac{D}{6} \right) \right]^{-1} (x^2 + 1)$$

$$= -\frac{1}{6D} \left[ 1 - \left( \frac{D}{6} - \frac{D^2}{6} \right) + \left( \frac{D}{6} - \frac{D^2}{6} \right)^2 \right]^{-1} (x^2 + 1)$$

$$= -\frac{1}{6D} \left[ x^2 + 1 - \left( \frac{D}{6} - \frac{D^2}{6} \right) x^2 + 1 + \left( \frac{D}{6} - \frac{D^2}{6} \right)^2 (x^2 + 1) \right]$$

$$= -\frac{1}{6D} \left[ x^2 + 1 - \frac{x^2}{6} + \frac{2}{6} + \frac{1}{36} (2) \right]$$

$$= -\frac{1}{6D} \left[ x^2 + 1 - \frac{x}{3} + \frac{1}{3} + \frac{1}{18} \right]$$

$$= -\frac{1}{6D} \left[ x^2 - \frac{x}{3} + \frac{28}{18} \right] = -\frac{1}{6} \left[ \frac{x^3}{3} - \frac{x^2}{6} + \frac{25}{18} \right]$$

$$P.I. = -\frac{x^3}{18} + \frac{x^2}{36} + \frac{28x}{108}$$

$$Y = C P^{-1} P.I.$$

$$= C_1 + C_2 e^{-2x} + C_3 e^{3x} - \frac{x^3}{18} + \frac{x^2}{36} + \frac{25x}{108}$$

Q.

$$(9) \text{ Solve } (D^2 - 2D + 5)y = \sin 3x.$$

Solu<sup>n</sup>:- The given differential equ<sup>n</sup> :-  
 $(D^2 - 2D + 5)y = \sin 3x.$

$\therefore$  The A.E is  $m^2 - 2m + 5 = 0$   
 $m = 1 \pm 2i$

$$C.F = e^x (C_1 \cos 2x + C_2 \sin 2x)$$

$$P.I = \frac{1}{(D^2 - 2D + 5)} \sin 3x = \frac{\sin 3x}{-(D^2 - 2D + 5)}$$

(Replace  $D^2$  by  $(3^2)$ )

$$= \frac{-1}{-2(D+2)} \sin 3x = -\frac{1}{2} (D-2) \cdot \frac{1}{(D^2-4)} \sin 3x$$

$$= -\frac{1}{2} (D-2) \cdot \frac{1}{-9-4} \sin 3x$$

$$= +\frac{1}{26} (D-2) \sin 3x = \frac{1}{26} (3 \cos 3x - 2 \sin 2x)$$

Hence solu<sup>n</sup> is

$$y = e^x (C_1 \cos 2x + C_2 \sin 2x) + \frac{1}{26} (3 \cos 3x - 2 \sin 2x)$$

$$(10) \text{ Solve } \frac{d^2y}{dx^2} + 3 \frac{dy}{dx} + 2y = e^{2x} \sin x.$$

Solu<sup>n</sup>:- The given equ<sup>n</sup> :-  
 $(D^2 + 3D + 2)y = e^{2x} \sin x$

$\therefore$  The A.E is  $(m^2 + 3m + 2) = 0$   
 $(m+1)(m+2) = 0$

$\therefore$  The roots are  $m = -1, -2$

$$C.F = C_1 e^{-x} + C_2 e^{-2x}$$

$$\therefore P.I = \frac{1}{D^2 + 3D + 2} e^{2x} \cdot \sin x$$

$$= e^{2x} \cdot \frac{1}{(D+2)^2 + 3(D+2) + 2} \sin x$$

$$\begin{aligned}
 &= e^{2x} \frac{1}{D^2+7D+12} \sin x = e^{2x} \frac{1}{-(1)^2+7D+12} \sin x \\
 &= e^{2x} \cdot \frac{1}{D+11} \sin x = e^{2x} (7D+11) \frac{1}{49D^2-121} \sin x \\
 &= e^{2x} (7D+11) \frac{1}{49(-1)-121} \sin x = \frac{e^{2x}}{-170} (7\cos x + 11\sin x) \\
 \therefore \text{Solu}^n \text{ is } y = C_1 e^{-x} + C_2 e^{-2x} - \frac{e^{2x}}{170} (7\cos x + 11\sin x)
 \end{aligned}$$

Case IV :- Evaluate:  $\frac{1}{D^2+a^2} \sin ax$

pt:- This we cannot evaluate by writing  $-a^2$  for  $D^2$ , as this given zero in denominator.  
 $\therefore$  we shall adopt different method by using the formula  $e^{i\theta} = \cos \theta + i \sin \theta$ .

$\therefore \sin ax = \text{Imaginary part of } e^{i ax}$

$$\begin{aligned}
 \therefore \frac{1}{D^2+a^2} \sin ax &= \text{Imaginary part of } \frac{1}{D^2+a^2} e^{i ax} \\
 &= \text{Img} \frac{1}{(D-ia)(D+ia)} e^{i ax} \\
 &= \text{Img} \frac{1}{(2ai)} \cdot \frac{x}{1!} e^{i ax} \\
 &= \text{Img} \frac{x}{2ai} (\cos ax + i \sin ax) \\
 &= \text{Imag} \frac{x}{2ai} (\cos ax + i \sin ax) \\
 &= \text{Imag} \frac{x}{2a} (-i \cos ax + \sin ax) \\
 &= -\frac{x}{2a} \cos ax \\
 &= -\frac{x}{2a} \cos ax
 \end{aligned}$$

Case V: Evaluate  $\frac{1}{D^2 + a^2} \cos ax$ .

∴ Same as above problem

$\frac{1}{D^2 + a^2} \cos ax = \text{Real part of } \frac{1}{D^2 + a^2} e^{iax}$

$$\frac{1}{D^2 + a^2} \cos ax = \frac{x}{2a} \sin ax.$$

Example: (1) Solve  $(D^2 + 4)y = \sin^2 x$

Solu<sup>n</sup>: The given D.E is

∴ The A.E is  $m^2 + 4 = 0$

$$m = \pm 2i$$

$$P.I = \frac{1}{D^2 + 4} \left[ \frac{1}{2} (1 - \cos 2x) \right]$$

$$= \frac{1}{2} \frac{1}{D^2 + 4} \cdot 1 = \frac{1}{2} \cdot \frac{1}{D^2 + 4} \cos 2x$$

$$= \frac{1}{2} \frac{1}{D^2 + 4} e^{ix} + \frac{1}{2} \frac{1}{D^2 + 4} \cos 2x$$

$$= \frac{1}{2} \frac{1}{4} \left( 1 - \frac{x}{4} \right) \sin 2x \quad (\because \frac{1}{D^2 + a^2} \cos ax = \frac{x}{2a} \sin ax)$$

$$= \frac{1}{8} \left( 1 - \frac{x}{4} \right) \sin 2x$$

$$P.I = \frac{1}{8} - \frac{x}{8} \sin 2x$$

$$\therefore \text{Solu}^n \quad y = c_1 \cos 2x + \frac{1}{2} \sin 2x + \frac{1}{8} - \frac{x}{8} \sin 2x.$$

(16)

$$② \text{ Solve: } (D^3 - 3D^2 + 9D - 27)y = \cos 3x$$

Solu<sup>n</sup>: - The given equ<sup>n</sup> is

$$(D^3 - 3D^2 + 9D - 27)y = \cos 3x$$

$$\therefore \text{The A.E is } (m^3 - 3m^2 + 9m - 27) = 0$$

$$(m-3)(m^2+9) = 0$$

$\therefore$  The roots are 3,  $3i$ ,  $-3i$

$$CF = C_1 e^{3x} + C_2 \cos 3x + C_3 \sin 3x$$

$$\therefore P.I = \frac{1}{(D^3 - 3D^2 + 9D - 27)} \cos 3x$$

Here denominator giving zero, when  $D^2$  is replaced by  $-3^2$

$$P.I = \frac{1}{(D^2+9)(D-3)}$$

$$= \frac{1}{(D^2+9)} \frac{(D+3)}{(D-3)} \frac{1}{(D^2+9)} \cos 3x$$

$$= -\frac{1}{8} \frac{1}{(D^2+9)} (-3 \sin 3x + 3 \cos 3x)$$

$$= -\frac{1}{8} \frac{1}{(D^2+9)} (\cos 3x - \sin 3x)$$

$$= -\frac{1}{6} \frac{\cos 3x}{D^2+9}, \quad \frac{1}{D^2+9} \sin 3x = -\frac{x}{6} \cos 3x$$

$$\therefore \frac{1}{D^2+9} \cos 3x = \frac{x}{6} \sin 3x, \quad \frac{1}{D^2+9}$$

$$P.I = -\frac{x}{36} (\sin 3x + \cos 3x)$$

Hence sol<sup>n</sup> is

$$y = C_1 e^{3x} + C_2 \cos 3x + C_3 \sin 3x - \frac{x}{36} (\sin 3x + \cos 3x)$$

Case VI:- To evaluate  $\frac{1}{f(D)} x^m$ , where  
 $m$  is +ve integer.

working Rule.

1. Take out of the lowest degree term from  $f(D)$  to make the first term unity,  
(so that Binomial theorem for a negative  
index is applicable)  
The remaining factor will be of the form  
 $[1 + \phi(D)]$  or  $[1 - \phi(D)]$   
2. Take this factor in numerator, It takes the  
 $[1 + \phi(D)]^l$  or  $[1 - \phi(D)]^l$
2. Take this factor in numerator, It takes the  
Binomial theorem
3. Expand it by Binomial theorem  
 $(1+x)^l = 1+x+x^2+x^3+\dots$   
 $(1-x)^l = 1-x+x^2-x^3+\dots$  into the

(17)

Example. ①  $(D^2 + D - 6)y = x$

Solu<sup>n</sup>: Given equ<sup>n</sup> is  $(D^2 + D - 6)y = x$

A.B is  $(D^2 + D - 6) = 0$

$$(D+3)(D-2) = 0 \therefore D = -3, 2$$

$$C.F = C_1 e^{-3x} + C_2 e^{2x}$$

$$P.I = \frac{1}{D^2 + D - 6} x = \frac{1}{-6 \left[ 1 - \frac{D}{6} - \frac{D^2}{6} \right]} x$$

$$= -\frac{1}{6} \left[ 1 - \left( \frac{D}{6} + \frac{D^2}{6} \right) \right]^{-1} x$$

$$= -\frac{1}{6} \left[ 1 + \left( \frac{D}{6} + \frac{D^2}{6} \right) + \dots \right] x = -\frac{1}{6} \left[ x + \frac{1}{6} D(x) \right]$$

$$= -\frac{1}{6} \left[ x + \frac{x}{6} \right]$$

The complete sol<sup>n</sup> is  
 $y = C_1 e^{-3x} + C_2 e^{2x} - \frac{1}{36} (6x + 1)$

② Solve  $(D^3 - 3D - 2)y = x^2$ .

Given equ<sup>n</sup> is  $(D^3 - 3D - 2) = 0$

A.B  $D^3 - 3D - 2 = 0$

put  $D = -1$ , L.H.S A.B = 0

$(D+1)$  is a factor.

$$\begin{array}{r} 1 & 0 & -3 & -2 \\ \times & & 1 & 2 \\ \hline & 1 & -1 & -2 & 0 \end{array}$$

$$\therefore (D+1)(D^2 - D - 2) = 0$$

$$(D+1)(D-2)(D+1) = 0$$

$$D = -1, 1, 2$$

$$C.F = (C_1 + C_2 x) e^{-x} + C_3 e^{2x} x^2$$

$$P.I = \frac{1}{D^3 - 3D - 2} x^2 = \frac{1}{-2 \left[ 1 + \frac{3D}{2} - \frac{D^3}{2} \right]} x^2$$

$$= -\frac{1}{2} \left[ 1 + \left( \frac{3D}{2} - \frac{D^3}{2} \right) \right]^{-1} x^2$$

$$= -\frac{1}{2} \left[ 1 - \left( \frac{3D}{2} - \frac{D^3}{2} \right) + \left( \frac{3D}{2} - \frac{D^3}{2} \right)^2 - \dots \right] x^2$$

( $\because$  expansion is to be carried upto term containing  $D^2$  only)

$$= -\frac{1}{2} \left[ 1 - \frac{3D}{2} + \frac{9D^2}{4} + \dots \right] x^2 = -\frac{1}{2} \left[ 1 - \frac{3}{2} D(x^2) + \frac{9}{4} D^2(x^2) \right]$$

$$= -\frac{1}{2} \left[ x^2 - \frac{3}{2} \cdot 2x + \frac{9}{4} \cdot 2 \right] = -\frac{1}{2} \left[ x^2 - 3x + \frac{9}{2} \right]$$

The complete solution is

$$y = C_1 + C_2 x + C_3 e^{-x} + \frac{1}{2} (x^2 - 3x + \frac{9}{2})$$


---

$$(3) \text{ Ques } \frac{d^3y}{dx^3} - 13 \frac{dy}{dx} + 12y = x^2$$

$$\text{Solu.:- } A, B \quad D^3 - 13D + 12 = 0$$

$$D(D^2 - 1) - 12(D - 1) = 0$$

$$(D-1)[D(D+1) - 12] = 0$$

$$(D-1)(D^2 + D - 12) = 0$$

$$(D-1)(D+4)(D-3) = 0$$

$$D = 1, -4, +3$$

$$CF = C_1 e^x + C_2 e^{-4x} + C_3 e^{3x}$$

$$P.F = \frac{1}{D^3 - 13D + 12} x^2$$

$$= \frac{1}{12} \left[ \frac{1}{1 - \frac{13}{12}D + \frac{D^3}{12}} \right] x^2 = \frac{1}{12} \left[ 1 + \left( \frac{13}{12}D - \frac{D^3}{12} \right) \right] x^2$$

$$= \frac{1}{12} \left[ 1 + \left( \frac{13}{12}D + \frac{-D^3}{12} \right) + \dots \right] x^2$$

$$= \frac{1}{12} \left[ x^2 + \frac{13}{12} D(x^2) + -\frac{D^3}{12} (x^2) \right]$$

$$= \frac{1}{12} \left[ x^2 - \frac{13}{12} \times 2x - 0 \right] = \frac{x^2}{12} - \frac{13}{72} x$$

1

(+) Solve  $(D^2 - 4)y = x^2$ .

A.B is  $D^2 - 4 \neq 0$ ,  $D = \pm 2$

$$CF = C_1 e^{2x} + C_2 e^{-2x}$$

$$P.I = \frac{1}{(D^2 - 4)} x^2 = \frac{1}{-4(1 - \frac{D^2}{4})} x^2$$

$$= -\frac{1}{4} \left[ 1 - \frac{D^2}{4} \right]^{-1} x^2 = -\frac{1}{4} \left[ 1 + \frac{D^2}{4} + \dots \right] x^2$$

$$= -\frac{1}{4} \left[ x^2 + \frac{1}{4} D^2 (x^2) \right] = -\frac{1}{4} (x^2 + \frac{1}{4} x^2)$$

$$= -\frac{1}{4} (x^2 + \frac{1}{2} x^2)$$

$\therefore$  The complete solution

$$y = C_1 e^{2x} + C_2 e^{-2x} + -\frac{1}{4} (x^2 + \frac{1}{2} x^2)$$


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Case(VII):- To evaluate  $\frac{1}{f(D)}(e^{ax} v)$ , where  
 $v$  is a function of  $x$

$$\frac{1}{f(D)}(e^{ax} v) = e^{ax} \frac{1}{f(D+a)} v$$

Pf:- Let  $x$  be a function of  $n$ . By successive differentiation we have

$$D(e^{ax} x) = e^{ax} Dx + a e^{ax} x$$

$$= e^{ax} [D+a] x$$

$$D^2(e^{ax} x) = e^{ax} D^2 x + a e^{ax} Dx + a^2 e^{ax} x$$

$$= e^{ax} (D^2 + 2aD + a^2) x = e^{ax} (D+a^2) x$$

$$\text{III } \therefore D^3(e^{ax} x) = e^{ax} (D+a)^3 x$$

$$\overbrace{D^n(e^{ax} x)} = e^{ax} \overbrace{(D+a)^n x} \quad \textcircled{1}$$

$$f(D) e^{ax} x = e^{ax} f(D+a) x \quad \textcircled{2}$$

$$\therefore \frac{1}{f(D+a)} v = x \quad \textcircled{3}$$

$$\text{from (1)} \quad f(D) \left[ e^{ax} \frac{1}{f(D+a)} v \right] = e^{ax} v$$

$$e^{ax} \frac{1}{f(D+a)} v = \frac{1}{f(D)} (e^{ax} v)$$

$$\therefore \boxed{\frac{1}{f(D)} (e^{ax} v) = e^{ax} \frac{1}{f(D+a)} v}$$

Examp[les:- ① Solve  $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = e^x \cos x$  (17)

Solu<sup>n</sup>:- A.B  $D^2 - 2D + 4 = 0$

$$D = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm 2i\sqrt{3}}{2} = 1 \pm i\sqrt{3}$$

$$CF = e^x (C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x)$$

$$P.I = \frac{1}{D^2 - 2D + 4} (e^x \cos x) = e^x \cdot \frac{1}{(D+1)^2 - 2(D+1) + 4} \cos x$$

$$= e^x \cdot \frac{1}{(D^2+3)} \cos x = e^x \frac{1}{-1+3} \cos x = \frac{1}{2} e^x \cos x$$

∴ The complete solu<sup>n</sup>:

$$y = e^x (C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x) + \frac{1}{2} e^x \cos x$$

② Solve  $(D^2 - 5D + 6)y = x e^{4x}$

Solu<sup>n</sup>:- A.B  $(D^2 - 5D + 6) = 0$

$$(D-2)(D-3) = 0 \quad \therefore D = 2, 3$$

$$CF = C_1 e^{2x} + C_2 e^{3x}$$

$$P.I = \frac{1}{D^2 - 5D + 6} (x e^{4x}) = \frac{1}{(D+4)^2 - 5(D+4) + 6} x e^{4x}$$

$$= e^{4x} \cdot \frac{1}{(D^2 + 3D + 2)} x = e^{4x} \frac{1}{2(D + \frac{3D}{2} + \frac{D^2}{2})} x$$

$$= \frac{1}{2} e^{4x} \left[ 1 + \left( \frac{3D}{2} + \frac{D^2}{2} \right) \right]^{-1} x$$

$$= \frac{1}{2} e^{4x} \left[ 1 - \left( \frac{3D}{2} + \frac{D^2}{2} \right) x - \dots \right] x$$

$$= \frac{1}{2} e^{4x} \left[ 1 - \frac{3D}{2} x - \dots \right] x = \frac{1}{2} e^{4x} \left[ x - \frac{3}{2} D(x) \right]$$

$$= \frac{1}{2} e^{4x} \left( x - \frac{3}{2} \right)$$

∴ The complete solu<sup>n</sup>:

$$y = C_1 e^{2x} + C_2 e^{3x} + \frac{1}{2} e^{4x} \left( x - \frac{3}{2} \right)$$

B.L. (19)

Example:- (1) Solve  $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = e^x \cos x$

Solu<sup>n</sup>:- A.B  $D^2 - 2D + 4 = 0$

$$D = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm 2i\sqrt{3}}{2} = 1 \pm i\sqrt{3}$$

$$CF = e^x (C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x)$$

$$P.I. = \frac{1}{D^2 - 2D + 4} (e^x \cos x) = e^x \cdot \frac{1}{(D+1)^2 - 2(D+1) + 4} \cos x$$

$$= e^x \cdot \frac{1}{(D^2+3)} \cos x = e^x \frac{1}{-1+3} \cos x = \frac{1}{2} e^x \cos x$$

∴ The complete solu<sup>n</sup>:

$$y = e^x (C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x) + \frac{1}{2} e^x \cos x$$

(2) Solve  $(D^2 - 5D + 6)y = x e^{4x}$

Solu<sup>n</sup>:- A.B  $(D^2 - 5D + 6) = 0$   
 $(D-2)(D-3) = 0 \quad \therefore D = 2, 3$

$$CF = C_1 e^{2x} + C_2 e^{3x}$$

$$P.I. = \frac{1}{D^2 - 5D + 6} (x e^{4x}) = \frac{1}{(D+4)^2 - 5(D+4) + 6} x e^{4x}$$

$$= e^{4x} \cdot \frac{1}{(D^2 + 3D + 2)} x = e^{4x} \frac{1}{2(D+1) + \frac{D^2}{2}} x$$

$$= \frac{1}{2} e^{4x} \left[ 1 + \left( \frac{3D}{2} + \frac{D^2}{2} \right) \right] x$$

$$= \frac{1}{2} e^{4x} \left[ 1 - \left( \frac{3D}{2} + \frac{D^2}{2} \right) x - \dots \right] x$$

$$= \frac{1}{2} e^{4x} \left[ 1 - \frac{3D}{2} x - \dots \right] x = \frac{1}{2} e^{4x} \left[ x - \frac{3}{2} D(x) \right]$$

$$= \frac{1}{2} e^{4x} \left( x - \frac{3}{2} \right)$$

∴ The complete solu<sup>n</sup>:

$$y = C_1 e^{2x} + C_2 e^{3x} + \frac{1}{2} e^{4x} \left( x - \frac{3}{2} \right)$$

③ Solve  $(D^2 + 1)y = xe^{2x}$ .

A.E is  $D^2 + 1 = 0$ ,  $\therefore D = \pm i$

$$CF = C_1 \cos x + C_2 \sin x$$

$$P.I = \frac{1}{(D^2 + 1)} (xe^{2x}) = e^{2x} \cdot \frac{x}{(D+2)^2 + 1}$$

$$= e^{2x} \cdot \frac{1}{D^2 + 4D + 5} x = e^{2x} \frac{1}{5(1 + \frac{4D}{5} + \frac{D^2}{5})} x$$

$$= \int e^{2x} \left[ 1 + \left( \frac{4D}{5} + \frac{D^2}{5} \right) \right] dx$$

$$= \int e^{2x} \left[ 1 - \left( \frac{4D}{5} + \frac{D^2}{5} \right) + \dots \right] dx$$

$$= \int \frac{1}{5} e^{2x} \left[ x - \frac{4}{5} D(x) \right] = \int \frac{1}{5} e^{2x} \left( x - \frac{4}{5} x \right)$$

The complete soln is

$$y = C_1 \cos x + C_2 \sin x + \frac{1}{5} e^{2x} \left( x - \frac{4}{5} x \right)$$

④ solve  $(D^2 + D - 2)y = e^x \sin nx$

A.E is  $D^2 + D - 2 = 0$ ,

$$(D+2)(D-1) = 0$$

$$D = 1, -2$$

$$CF = C_1 e^x + C_2 e^{-2x}$$

$$P.I = \frac{1}{D^2 + D - 2} (e^x \sin nx) = e^x \cdot \frac{1}{(D+1)^2 + (D+1) - 2} \sin nx$$

$$= e^x \cdot \frac{1}{D^2 + 3D} \sin nx = e^x \cdot \frac{1}{-1^2 + 3D} \sin nx = \frac{e^x (3D+1)}{(-1^2 + 3D)(-1 + 3D)} \sin nx$$

$$= e^x \cdot \frac{3D+1}{9D^2 - 1} \sin nx = e^x \cdot \frac{3D+1}{9(-1)^2 - 1} \sin nx$$

$$= -\frac{1}{10} e^x (3 \cos x + \sin x)$$

$\therefore$  The complete soln is

$$y = C_1 e^x + C_2 e^{-2x} - \frac{1}{10} e^x (3 \cos x + \sin x)$$

Case IX :- P.T with usual notation

$$\frac{1}{f(D)}(xv) = \left\{ x - \frac{f'(D)}{f(D)} \right\} v \text{ where } v \text{ is a function of } x$$

Pf:- By successive differentiation.

Let  $v_1$  be a function of  $x$ .

$$D(xv_1) = xDv_1 + v_1$$

$$\begin{aligned} D^2(xv_1) &= D(xDv_1 + v_1) \\ &= xD^2v_1 + Dv_1 + DV_1 + xD^2v_1 + 2DV_1 \\ &= DV_1(xDv_1 + 2) \end{aligned}$$

$$\begin{aligned} D^3(xv_1) &= D[xD^2v_1 + xDV_1] \\ &= xD^3v_1 + D^2v_1 + 2D^2v_1 \\ &= xD^3v_1 + 3D^2v_1 \end{aligned}$$

$$\begin{aligned} D^n(xv_1) &= xD^n v_1 + D^{n-1} v_1 \\ &= xD^n v_1 + \left[ \frac{d}{dx} D^n(D^n v_1) \right] \end{aligned}$$

$$D^n(xv_1) = xD^n v_1 + f'(D)v_1$$

$f'(D)(xv_1) = xf(D)v_1 + f'(D)v_1$  ①  
 $v_1$  is a function of  $x$ ,  $v$  is a function of  $x$

$$f(D)v_1 = v$$

$$\begin{aligned} v_1 &= \frac{1}{f(D)}v \quad \text{sub in ①} \\ \frac{1}{f(D)}[f(D)xv] &= x \frac{1}{f(D)} [f(D)v] + \frac{1}{f(D)} [f'(D)v] \end{aligned}$$

$$\frac{1}{f(D)}[f(D)xv] = x \frac{1}{f(D)} v + \frac{1}{f(D)} [f'(D)v]$$

Operating  $\frac{1}{f(D)}$  both sides

$$x \frac{1}{f(D)} v = \frac{1}{f(D)}(xv) + \frac{1}{f(D)} \left[ \frac{f'(D)}{f(D)} \right] v$$

$$\frac{1}{f(D)}(xv) = \left\{ x - \frac{1}{f(D)}v - \frac{1}{f(D)} \left[ \frac{f'(D)}{f(D)} \right] v \right\}$$

$$= \left[ \frac{1}{f(D)} (xv) + \left\{ x - \frac{f'(D)}{f(D)} \right\} \frac{1}{f(D)} v \right]$$

OR

$$\frac{1}{f(D)} (xv) = \frac{x}{f(D)} v - \frac{f'(D)}{[f(D)]^2} v^2$$

Example.

$$(1) \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = x \sin x$$

Soln.  $(D^2 - 2D + 1)y = x \sin x.$

A.E is  $D^2 - 2D + 1 = 0$   
 $(D-1)^2 = 0, D = 1, 1$

$$CF = (C_1 + C_2 x)e^x$$

$$P.I = \frac{1}{(D^2 - 2D + 1)} \sin x + \int_D \left( \frac{1}{D^2 - 2D + 1} \right) \sin x$$

$$= x \cdot \frac{1}{(D^2 - 2D + 1)} \sin x$$

$$= x \cdot \frac{1}{D^2 - 2D + 1} \sin x - \frac{2D - 2}{(D^2 - 2D + 1)^2} \sin x$$

$$= x \cdot \frac{1}{-1 - 2D + 1} \sin x - \frac{2(D-1)}{(-1 - 2D + 1)^2} \sin x$$

$$= x \cdot \frac{1}{D} \sin x - \frac{(D-1)}{2D^2} \sin x$$

$$= x \cdot \int \sin x dx - \frac{D-1}{2(D-1)} \sin x$$

$$= -\frac{x}{2} \cos x + \frac{1}{2} [D(\sin x) - \sin x]$$

$$= -\frac{x}{2} \cos x + \frac{1}{2} (x \cos x - \sin x)$$

: The complete soln is

$$y = (C_1 + C_2 x)e^x + \frac{x}{2} \cos x + \frac{1}{2} (\cos x - \sin x)$$

(24)

② Solve  $\frac{d^2y}{dx^2} + 4y = x \sin x$

P.T.: -  $(D^2 + 4)y = \sin x$   
A.E is  $D^2 + 4 = 0 \Rightarrow D = \pm 2i$

$$CF = C_1 \cos 2x + C_2 \sin 2x$$

$$P.I = \frac{1}{(D^2+4)} x \sin x = x \cdot \frac{1}{(D^2+4)} \sin x = \frac{x \sin x - \frac{2}{4} \sin x}{(D^2+4)^2}$$

$$\left( \because \frac{1}{f(D)}(xv) = x \cdot \frac{1}{f(D)} v - \frac{f'(D)}{(f(D))^2} v \right)$$

$$= x \cdot \frac{1}{-4} \sin x - \frac{2}{4} \sin x$$

$$= \frac{1}{3} x \sin x - \frac{2}{4} D(\sin x) = \frac{1}{3} x \sin x - \frac{2}{4} \cos x$$

The complete soln is

$$y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{3} x \sin x - \frac{2}{4} \cos x$$

③ Solve  $\frac{d^2y}{dx^2} - 4y = x \cos 2x$

Soln: -  $(D^2 - 4)y = x \cos 2x$

A.E is  $D^2 - 4 = 0 \Rightarrow D = \pm 2$

$$CF = C_1 e^{2x} + C_2 e^{-2x}$$

$$P.I = \frac{1}{(D^2-4)} (x \cos 2x) = x \cdot \frac{1}{(D^2-4)} \cos 2x - \frac{2}{(D^2-4)^2} \cos 2x$$

$$\left( \text{using } \frac{1}{f(D)}(xv) = x \cdot \frac{1}{f(D)} v - \frac{f'(D)}{(f(D))^2} v \right)$$

$$= x \cdot \frac{1}{-4} \cos 2x - \frac{2}{(-4-4)^2} \cos 2x$$

$$= -\frac{1}{8} x \cos 2x - \frac{1}{32} D(\cos 2x)$$

$$= -\frac{1}{8} x \cos 2x - \frac{1}{32} (-2 \sin 2x) = -\frac{1}{8} x \cos 2x + \frac{1}{16} \sin 2x$$

The complete soln is

$$y = C_1 e^{2x} + C_2 e^{-2x} - \frac{1}{8} x \cos 2x + \frac{1}{16} \sin 2x$$

$$④ \text{ Solve } \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$$

Here C.F. =  $(C_1 + C_2 x) e^x$

$$\text{P.I.} = \frac{1}{(D-1)^2} (xe^x \sin x) = e^x \cdot \frac{1}{(D+1-1)^2} (x \sin x)$$

$$= e^x \cdot \frac{1}{D^2} (x \sin x)$$

$$= e^x \left[ x \cdot \frac{1}{D^2} \sin x + \frac{1}{D^2} (\cancel{x}) \sin x \right] \quad (\because \cancel{x} = \text{Integration})$$

$$= e^x \left[ x \frac{1}{D-1} \sin x - \frac{2}{D^3} \sin x \right]$$

$$= e^x \left[ -x \sin x - \frac{2}{D^2} (-\cos x) \right] = e^x (-x \sin x - 2 \cos x)$$

$$\text{The complete soln. is } y = (C_1 + C_2 x) e^x - e^x (x \sin x + 2 \cos x)$$

$$(4) \text{ Solve } \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$$

$$\text{Here } C.F = C_1 + C_2 x e^x$$

$$P.I = \frac{1}{(D-1)^2} (xe^x \sin x) = e^x \cdot \frac{1}{(D+1-1)^2} (x \sin x)$$

$$= e^x \cdot \frac{1}{D^2} (x \sin x)$$

$$= e^x \left[ x \cdot \frac{1}{D^2} \sin x + \frac{1}{D^2} (\int x \sin x) \right] \quad (\because \int x \text{ is integration})$$

$$= e^x \left[ x \cdot \frac{1}{D-1} \sin x - \frac{2}{D^3} \sin x \right]$$

$$= e^x \left[ -x \sin x - \frac{2}{D^2} (-\cos x) \right]$$

$$= e^x \left[ -x \sin x + \frac{2}{D} \sin x \right] = e^x (-x \sin x + 2 \cos x)$$

The complete solution is

$$y = (C_1 + C_2 x) e^x - e^x (x \sin x + 2 \cos x)$$

## Unit II.

### Homogeneous Linear Equation.

#### I. Homogeneous Linear Equation.

Definition:- An equation of the form,

$$a_0 x^n \frac{dy}{dx^n} + a_1 x^{n-1} \frac{dy}{dx^{n-1}} + a_2 x^{n-2} \frac{dy}{dx^{n-2}} + \dots + a_{n-1} x \frac{dy}{dx} + a_n y = 0$$

$$+ a_n y = 0$$

where  $a_0, a_1, a_2, \dots, a_n$  are constants &  $x$  is a function of  $x$  is called homogeneous linear equation of  $n^{\text{th}}$  order.

#### II Method of Solution.

##### I Method of Solution.

Reduce the homogeneous linear equation.

$$a_0 x^n \frac{dy}{dx^n} + a_1 x^{n-1} \frac{dy}{dx^{n-1}} + \dots + a_{n-1} x \frac{dy}{dx} + a_n y = 0$$

into linear equation with constant coefficient if

pt:- The given homogeneous linear equation is

$$a_0 x^n \frac{dy}{dx^n} + a_1 x^{n-1} \frac{dy}{dx^{n-1}} + a_2 x^{n-2} \frac{dy}{dx^{n-2}} + \dots + a_{n-1} x \frac{dy}{dx} + a_n y = 0 \quad (1)$$

$$\text{Put } x = e^z$$

$$z = \log x \quad \therefore \frac{dz}{dx} = \frac{1}{x}$$

$$\therefore \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz} = x \frac{dy}{dz} = \frac{dy}{dz} \quad (2)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \left( \frac{dy}{dz} \right) = \frac{1}{x} \frac{d^2y}{dz^2} \cdot \frac{dz}{dx} - \frac{1}{x^2} \cdot \frac{dy}{dz} \quad (\because \frac{dz}{dx} = \frac{1}{x})$$

$$= \frac{1}{x^2} \left( \frac{d^2y}{dz^2} - \frac{dy}{dz} \right)$$

$$\Rightarrow x^2 \cdot \frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$$

$$\text{put } x \cdot \frac{d}{dx} = \frac{d}{dz} = D.$$

$$\text{From (2)} \quad x \cdot \frac{dy}{dx} = Dy$$

$$\text{From (3)} \quad x^2 \frac{d^2y}{dx^2} = D^2y - Dy = D(D-1)y$$

$$\text{Hence } x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$$

$$x^n \frac{d^ny}{dx^n} = D(D-1)(D-2) \dots (D-n+1)y$$

$\therefore$  From (1) we have.

$$[a_0 D(D-1) \dots (D-n+1) + a_1 D(D-1) \dots (D-n+2) + \dots + a_{n-2} D(D-1) + a_{n-1} D + a_n]y = f(e^z)$$

where  $x = f(z)$ .

This is linear eqn with constant coefficients  
it is therefore solvable for  $y$  in terms of  $z$ .

If  $y = F(z)$  is its solution,

$$\text{put } z = \log x.$$

$\therefore$  The required solution is

$$y = F(\log x) \quad (\because z = \log x)$$

---

NOTE:- Put  $x = e^z$

$$D = x \cdot \frac{d}{dx} = \frac{d}{dz}$$

$$x^2 \frac{d^2y}{dx^2} = D(D-1)$$

$$x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2) \dots$$

Given eqn becomes

$$f(z)y = z. \quad (\text{A function of } z)$$

Solve it & replace  $z$  by  $\log x$

$$\text{Solve. } (1) \frac{x^2 d^2 y}{dx^2} + x \frac{dy}{dx} - 9y = 0$$

Solu<sup>n</sup>: Put  $x = e^z$  so that  $z = \log x$

$$D = \frac{d}{dz} = x \frac{d}{dx} \text{ given equn becomes}$$

$$[D(D-1) + D - 9]y = 0 \text{ or } (D^2 - 9)y = 0$$

$$\text{A.R is } D^2 - 9 = 0$$

$$\therefore D = \pm 3$$

$$\therefore y = C_1 e^{3z} + C_2 e^{-3z}$$

$$= C_1 x^3 + C_2 x^{-3}$$


---

$$(2) \text{ Solve } \frac{x^2 d^2 y}{dx^2} + 3x \frac{dy}{dx} + y = 0 \quad (1)$$

Solu<sup>n</sup>: Put  $x = e^z$  so that  $z = \log x$

let  $D = \frac{d}{dz} = x \cdot \frac{d}{dx}$  The given equn (1) becomes

$$[D(D-1) + 3D + 1]y = 0 \text{ or } (D^2 + 2D + 1)y = 0$$

$$\text{or } (D+1)^2 y = 0$$

$$\text{A.R is } (D+1)^2 = 0 \quad \therefore D = -1, -1$$

$$\text{Complete soln } y = (C_1 + C_2 z) e^{-2} \text{ or}$$

$$y = (C_1 + C_2 \log x) x^{-1}$$

$$y = \frac{1}{x} (C_1 + C_2 \log x)$$


---

$$(3) \text{ Solve } \frac{x^2 d^2 y}{dx^2} + 2x \frac{dy}{dx} + 2y = 0$$

$\Rightarrow$  put  $x = e^z, z = \log x$

$$D = \frac{d}{dz} = x \cdot \frac{d}{dx}$$

$$[D(D-1) + 2D + 2]y = 0$$

$$(D^2 - D + 2D + 2)y = 0$$

$$(D^2 + D + 2)y = 0$$

$$\text{A.R } z^2 + z + 2 = 0 \Rightarrow z = \frac{-1 \pm \sqrt{1-4(1)(2)}}{2}$$

$$= \frac{-1 \pm \sqrt{7}}{2}$$

$$CF = e^{\frac{1}{2}x^2} \left[ c_1 \cos \frac{\sqrt{7}}{2}x + c_2 \sin \frac{\sqrt{7}}{2}x \right]$$

$$y = x^{\frac{1}{2}} \left[ c_1 \cos \left( \frac{\sqrt{7}}{2} \log x \right) + c_2 \sin \left( \frac{\sqrt{7}}{2} \log x \right) \right]$$

(4) Solve  $x^3 \frac{d^3 y}{dx^3} + 2x^2 \frac{d^2 y}{dx^2} + 2y = \log(x+1/x)$   $\quad \text{(i)}$

Soln:- Put  $x = e^z \Rightarrow z = \log x$

Let  $D = \frac{d}{dz} = x \cdot \frac{d}{dx} \therefore \text{Eqn (i) becomes}$

$$\begin{aligned} [D(D-1)(D-2) + 2D(D-1) + 2]y &= \log(e^z + e^{-z}) \\ (D^3 - D^2 + 2)y &= \log(e^z + e^{-z}) \end{aligned}$$

$$A.E \text{ is } (D^3 - D^2 + 2) = 0$$

$$\therefore D = -1$$

$\therefore$  The roots are  $D^2 - 2D + 2 = 0$

$$D = \frac{-2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

Roots of A.E  $-1, 1 \pm i$

$$CF = c_1 e^{-z} + e^z (c_2 \cos z + c_3 \sin z)$$

$$P.I = 10 \frac{1}{D^3 - D^2 + 2} e^z + 10 \frac{1}{D^3 - D^2 + 2} \frac{e^z}{z}$$

$$= 10 \frac{1}{1-1+2} e^z + 10 z \cdot \frac{1}{3z^2 - 2z} \frac{e^z}{z}$$

$$= 5e^z + 10z \cdot \frac{1}{3z^2 - 2z} \frac{e^z}{z}$$

$$= 5e^z + 2z \frac{e^z}{z}$$

Completed of  $10^2$  is

$$y = c_1 e^{-z} + e^z (c_2 \cos z + c_3 \sin z) + 5e^z + 2z e^z$$

$$= c_1 x^{-1} + x \left[ c_2 \cos(\log x) + c_3 \sin(\log x) \right] + 5x + 2 \log x$$

$$\begin{array}{r|rrrr} & 1 & -1 & 0 & 2 \\ & -1 & & -1 & -2 \\ \hline 1 & -2 & 2 & 0 \end{array}$$

(24)

(5)  $x^2 \frac{d^2y}{dx^2} - xy \frac{dy}{dx} - 3yz = x^2 \log x$  2019.

pf:- put  $x = e^z$  so  $z = \log x$

$D = \frac{d}{dz}$   $\therefore z \frac{d}{dx}$  The given equ<sup>n</sup> becomes  
 $[D(D-1) - D - 3]y = z \cdot e^{2z}$  or  $(D^2 - 2D - 3)y = ze^{2z}$

A.B  $D^2 - 2D - 3 = 0$   
 $(D-3)(D+1) = 0$   
 $D = 3, -1$   
 $CF = C_1 e^{3z} + C_2 e^{-z}$

P.I.  $= \frac{1}{(D^2 - 2D - 3)} z e^{2z}$   $\therefore f(D) = \frac{e^{az}}{f(D+1)}$

$$= e^{2z} \cdot \frac{1}{(D+2)^2 - 2(D+2) - 3} z$$

$$= e^{2z} \cdot \frac{1}{(D+2)^2 - 2(D+2) - 3} z = e^{2z} \frac{1}{-3\left(\frac{1-2D}{3} - \frac{D^2}{3}\right)} z$$

$$= -\frac{1}{3} e^{2z} \left[ 1 - \left( \frac{2D}{3} + \frac{D^2}{3} \right) \right]^{-\frac{1}{2}} z$$

$$= -\frac{1}{3} e^{2z} \left[ \left( \frac{1+2D}{3} \right)^2 \right]^{-\frac{1}{2}} = -\frac{1}{3} e^{2z} \in z + \frac{2}{3}$$

complete sol<sup>n</sup> is

$$y = C_1 e^{3z} + C_2 e^{-z} - \frac{1}{3} e^{2z} \left( z + \frac{2}{3} \right)$$

$$= C_1 x^3 + C_2 \frac{1}{x} - \frac{x^2}{3} (\log x + \frac{2}{3})$$

$\therefore$  Equ<sup>n</sup> becomes

$$8z^3 \frac{d^3y}{dz^3} + 2z \frac{dy}{dz} - 2y = 0$$

Put  $z = e^u$ :  $\therefore u = \log z$

$$z^3 D^3 = D_1(D_1-1)(D_1-2) \text{ or } zD = D_1$$

$$\text{Here } D = \frac{d}{du} \text{ & } D_1 = \frac{d}{du}$$

$$\text{we have } [8D_1(D_1-1)(D_1-2) + 2D_1 - 2] y = 0$$

$$(8D_1^3 - 24D_1^2 + 18D_1 - 2)y = 0$$

$$\therefore \text{The A.R is } 8m^3 - 24m^2 + 18m - 2 = 0$$

$$4m^3 - 12m^2 + 9m - 1 = 0$$

$$(m-1)(4m^2 - 8m + 1) = 0$$

$$\therefore \text{The roots are } m = 1, 1 \pm \sqrt{3}$$

$$\therefore \text{The CF} = C_1 e^u + C_2 e^{(1+\sqrt{3}/2)u} + C_3 e^{(1-\sqrt{3}/2)u}$$

$$y_1 = C_1 e^u + C_2 e^{(1+\sqrt{3}/2)u} + C_3 e^{(1-\sqrt{3}/2)u}$$

$$y_2 = C_1 z + C_2 z^{(1+\sqrt{3}/2)} + C_3 z^{(1-\sqrt{3}/2)}$$

$$y_2 = [C_1 + C_2 z^{\sqrt{3}/2} + C_3 z^{-\sqrt{3}/2}]$$

$$y_2 = (2u-1) \{ C_1 + C_2 (2x-1)^{\sqrt{3}/2} + C_3 (2x-1)^{-\sqrt{3}/2} \}$$

$$\textcircled{1} \quad \begin{array}{l} \text{2017} \\ x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + 9y = 3x^2 + \sin(3\log x) \end{array} \quad (25)$$

pt:- The given Equ<sup>n</sup> is

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + 9y = 3x^2 + \sin(3\log x)$$

$$(D_1^2 + D_1 + 9)y = 3x^2 + \sin(3\log x)$$

$$\text{put } x = e^z \text{ i.e } z = \log x$$

$$x^2 D^2 = D_1(D_1 - 1) \quad (\because D_1 = \frac{d}{dz})$$

$$x D = D_1$$

$\therefore$  Equ<sup>n</sup> becomes

$$[D_1(D_1 - 1) + D_1 + 9]y = 3e^{2z} + \sin(3z)$$

$$(D_1^2 + 9)y = 3e^{2z} + \sin(3z)$$

$\therefore A.B$  is  $m^2 + 9 = 0$

$$m = 3i, -3i$$

$$C.F = C_1 \cos 3z + C_2 \sin 3z$$

$$P.I. = \frac{1}{D_1^2 + 9} (3e^{2z} + \sin 3z)$$

$$= \frac{1}{(D_1^2 + 9)} 3e^{2z} + \frac{1}{(D_1^2 + 9)} \sin 3z$$

$$\frac{1}{D_1^2 + 9} 3e^{2z} = 3 \cdot \frac{1}{(4+9)} e^{2z} = \frac{3}{13} e^{2z}$$

$$\therefore \frac{1}{D_1^2 + 9} \sin 3z$$

$\therefore$  Here  $D_1^2 + 9 = 0$ , when  $D_1^2$  replaced by  $-(3)^2$

$$\therefore \frac{1}{D_1^2 + 9} \sin 3z = \text{Imaginary part } \left( \frac{1}{(D_1^2 + 9)} e^{3iz} \right)$$

$$\left( e^{3iz} \right) = \cos 3z + i \sin 3z$$

$$\frac{1}{D_1^2 + 9} e^{3iz} = \frac{1}{(D_1 - 3i)(D_1 + 3i)} e^{3iz}$$

$$= \frac{1}{6!} \times \frac{1}{1!} e^{3iz} = \frac{1}{6!} ( \cos 3z + i \sin 3z )$$

$$P.I = \frac{3}{13} e^{2z} - \frac{2}{6} \cos 3z$$

$$\therefore \text{solution is } y = c_1 \cos 3z + c_2 \sin 3z + \frac{3}{13} e^{2z} - \frac{2}{6} \cos 3z$$

$$y = c_1 \cos(\log x^3) + c_2 \sin(\log x^3) + \frac{3}{13} x^2 - \frac{\log x}{6} \cos(\log x^3)$$

## (26)

### Linear Equations Reducible to Homogeneous Linear Form.

Method of solution of linear differential equations reducible to homogeneous linear form.

Consider the equation,

$$a_0(a+bx)^n \frac{d^n y}{dx^n} + a_1(a+bx)^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \dots + a_n \frac{(a+bx)dy}{dx}$$
$$+ a_n y = f(x) \quad (1)$$

where  $a_0, a_1, a_2, \dots, a_n$  are constant.

Put  $a+bx = e^z$  so  $z = \log(a+bx)$

$$\frac{dz}{dx} = \frac{b}{a+bx}$$

put  $\frac{dy}{dx} = D$  Then from ②  $(a+bx) \frac{dy}{dx} = bDy$

from ③  $(a+bx)^2 \frac{d^2y}{dx^2} = b^2(D^2y - Dy)$   
 $\therefore = b^2 D(D-1)y$

11<sup>th</sup>  $(a+bx)^3 \frac{d^3y}{dx^3} = b^3 D(D-1)(D-2)y$   
 $\vdots$

$(a+bx)^n \frac{d^n y}{dx^n} = b^n D(D-1)(D-2) \dots (D-n+1)y$

∴ Equn ① becomes

$$[a + b^n D(D-1) \dots (D-n+1) + a_1 b^{n-1} D(D-1) \dots (D-n+2) \\ + \dots + a_{n-1} b^1 + a_n] y = f\left(\frac{e^z - a}{b}\right)$$

which is a linear equation with  
constant coefficient & solvable for y in terms  
of z.

$$\therefore y = F(z)$$

$$\therefore z = \log(a+bx)$$

∴ The Required Solu<sup>n</sup> is

$$y = F[\log(a+bx)]$$

NOTE:- put  $a+bx = e^z$

$$z = \log(a+bx)$$

∴  $\frac{dy}{dz} = D$  then  $(a+bx) \frac{dy}{dx} = bDy$

$$(a+bx)^2 \frac{d^2y}{dx^2} = b^2 D(D-1)y \text{ etc}$$

$$(1) \text{ Solve } (5+2x)^2 \frac{d^2y}{dx^2} - 6(5+2x) \frac{dy}{dx} + 8y = 0 \quad (1)$$

pf:- put  $5+2x = e^z \Rightarrow z = \log(5+2x)$

let  $\frac{dy}{dz} = D$ . The equn becomes (1)

$$\Rightarrow [D^2(D-1) - 6D + 8]y = 0 \quad \text{here } b=2$$

$$4(D^2 - 4D + 2)y = 0 \quad \text{or } (D^2 - 4D + 2)y = 0$$

$$\text{A.E } D^2 - 4D + 2 = 0 \text{ where}$$

$$D = \frac{4 \pm \sqrt{16-8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$$

$$\therefore \text{Complete soln is } y = C_1 e^{(2+\sqrt{2})z} + C_2 e^{(2-\sqrt{2})z} = C_1 (5+2x)^{\frac{2+\sqrt{2}}{2}} + C_2 (5+2x)^{\frac{2-\sqrt{2}}{2}}$$

$$(2) \text{ Solve } (1+x)^2 \frac{d^2y}{dx^2} + (1+x) \frac{dy}{dx} + y = 4 \cos \log(1+x) \quad (1)$$

pf:- put  $1+x = e^z \Rightarrow z = \log(1+x)$

let  $\frac{dy}{dz} = D$ .  $\therefore$  equn (1) becomes

$$[D(D-1) + D + 1]y = 4 \cos z \quad (\because b=1)$$

$$(D^2 + 1)y = 4 \cos z$$

$$\text{A.E is } D^2 + 1 = 0 \Rightarrow D = \pm i$$

$$CF = C_1 \cos z + C_2 \sin z$$

$$P.I = 4 \cdot \frac{1}{(D^2 + 1)} \cos z \quad \text{case failure}$$

$$= 4z \cdot \frac{1}{2i} \cos z = 2z \int \cos z dz = 2z \sin z$$

$\therefore$  Complete soln is

$$y = C_1 \cos z + C_2 \sin z + 2z \sin z$$

$$= C_1 \cos [\log(1+x)] + C_2 \sin [\log(1+x)] + 2 \log(1+x) \sin(\log(1+x))$$

$$(3) (3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2) \frac{dy}{dx} - 36y = 3x^2 + 4x + 1 \quad \text{(1)}$$

Solu<sup>n</sup>: - put  $3x+2 = e^z \Rightarrow z = \log(3x+2)$   
 $\therefore$  Eqn (1) becomes  $\frac{d^2y}{dz^2} = 0 \quad \therefore b^2 = 3$

$$[3^2 D(D-1) + 3 \cdot 3D - 36]y = 3\left(\frac{e^2-2}{3}\right)^2 + 4\left(\frac{e^2-2}{3}\right) + 1$$

$$\Rightarrow 9(D^2-1+D-4)y = \frac{1}{3}(e^{22}-4e^{22}+4) + \frac{4}{3}(e^{22}-2) + 1$$

$$9(D^2-4)y = \frac{1}{3}e^{22} - \frac{1}{3}$$

$\therefore A, B$  is  $9(D^2-4) = 0$  where  $D = \pm 2$

$$CF = C_1 e^{2z} + C_2 e^{-2z} \quad e^{2z} - \frac{1}{27} \cdot \frac{1}{(D^2-4)} e^{0z}$$

$$P.I = \frac{1}{9(D^2-4)} \left( \frac{1}{3} e^{22} - \frac{1}{3} \right) = \frac{1}{27} \frac{1}{(D^2-4)} e^{22} + \frac{1}{27} \cdot \frac{1}{(D^2-4)} e^{0z}$$

$$= \frac{1}{27} \cdot \frac{1}{20} e^{22} - \frac{1}{27} \cdot \frac{1}{8} e^{-22} = \frac{1}{54} \left\{ e^{22} \delta^2 + \frac{1}{108} \right\}$$

$$= \frac{1}{54} \cdot \frac{e^{22}}{2} + \frac{1}{108} = \frac{1}{108} (ze^{2z} + 1)$$

$\therefore$  complete solu<sup>n</sup> is

$$y = C_1 e^{2z} + C_2 e^{-2z} + \frac{1}{108} (ze^{2z} + 1)$$

$$= C_1 (3x+2)^2 + C_2 (3x+2)^{-2} + \frac{1}{108} ((3x+2)^2 \log(3x+2) + 1)$$

$$(3) (3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2) \frac{dy}{dx} - 3y = 3x^2 + 4x + 1 \quad \text{(1)}$$

Solu<sup>n</sup>: - put  $3x+2 = e^z \Rightarrow z = \log(3x+2)$   
 $\therefore$  Equ<sup>n</sup> (1) becomes  $\frac{d^2y}{dz^2} = 0 \quad \therefore b^2 = 3$

$$[3^2(D-1) + 3 \cdot 3D - 36]y = 3\left(\frac{e^2-2}{3}\right)^2 + 4\left(\frac{e^2-2}{3}\right) + 1 \\ \Rightarrow 9(D^2-1+D-4)y = \frac{1}{3}(e^{22}-4e^{22}+4) + \frac{4}{3}(e^{22}-2)+1 \\ 9(D^2-4)y = \frac{1}{3}e^{22}-\frac{1}{3}$$

$\therefore A, B$  is  $9(D^2-4)=0$  where  $D= \pm 2$

$$CF = C_1 e^{2z} + C_2 e^{-2z} \quad e^{2z} - \frac{1}{27} \cdot \frac{1}{(D^2-4)} e^{0z} \\ P.I = \frac{1}{9(D^2-4)} \left( \frac{1}{3} e^{2z} - \frac{1}{3} \right) = \frac{1}{27} \frac{1}{(D^2-4)} \\ = \frac{1}{27} \cdot \frac{1}{20} e^{2z} - \frac{1}{27} \cdot \frac{1}{0-4} = \frac{2}{54} \int e^{2z} dz + \frac{1}{108} \\ = \frac{1}{54} \cdot e^{2z} + \frac{1}{108} = \frac{1}{108} (ze^{2z} + 1)$$

$\therefore$  complete Solu<sup>n</sup> is

$$y = C_1 e^{2z} + C_2 e^{-2z} + \frac{1}{108} (ze^{2z} + 1)$$

$$= C_1 (3x+2)^2 + C_2 (3x+2)^{-2} + \frac{1}{108} ((3x+2)^2 \log(3x+2) + 1)$$

$$(4) \text{ Solve } (1+2x)^2 \frac{d^2y}{dx^2} + 6(1+2x) \frac{dy}{dx} + 16y = 0$$

Solu<sup>n</sup>: - Put  $z = 1+2x \Rightarrow dz = 2dx \Rightarrow dx = \frac{dz}{2}$

$$(3) (3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2) \frac{dy}{dx} - 36y = 3x^2 + 4x + 1 \quad (1)$$

Solu<sup>n</sup>: - Put  $3x+2 = e^z \Rightarrow z = \log(3x+2)$   
 $\therefore$  Eqn (1) becomes  $\frac{d^2y}{dz^2} = 0 \quad \because b^2 = 3$

$$[3^2(D-1) + 3 \cdot 3D - 36]y = 3\left(\frac{e^2-2}{3}\right)^3 + 4\left(\frac{e^2-2}{3}\right) + 1$$

$$\Rightarrow 9(D^2-1+D-4)y = \frac{1}{3}(e^{22}-4e^2+4) + \frac{4}{3}(e^2-2) + 1$$

$$9(D^2-4)y = \frac{1}{3}e^{22} - \frac{1}{3}$$

$$\therefore A.B \text{ is } 9(D^2-4) = 0 \text{ where } D = \pm 2$$

$$CF = C_1 e^{2z} + C_2 e^{-2z}$$

$$P.I = \frac{1}{9(D^2-4)} \left( \frac{1}{3}e^{22} - \frac{1}{3} \right) = \frac{1}{27} \frac{1}{(D^2-4)} e^{2z} - \frac{1}{27} \cdot \frac{1}{(D^2-4)} e^{0z}$$

$$= \frac{1}{27} \left[ \frac{1}{20} e^{2z} - \frac{1}{27} \cdot \frac{1}{0-4} \right] = \frac{1}{54} \int e^{2z} dz + \frac{1}{108}$$

$$= \frac{1}{54} \cdot \frac{e^{2z}}{2} + \frac{1}{108} = \frac{1}{108} (ze^{2z} + 1)$$

$\therefore$  complete Solu<sup>n</sup> is

$$y = C_1 e^{2z} + C_2 e^{-2z} + \frac{1}{108} (ze^{2z} + 1)$$

$$= C_1 (3x+2)^2 + C_2 (3x+2)^{-2} + \frac{1}{108} ((3x+2)^2 \log(3x+2) + 1)$$

$$(4) \text{ Solve } (1+2x)^2 \frac{\partial^2 y}{\partial x^2} - 6(1+2x) \frac{\partial y}{\partial x} + 16y = 8(1+2x)^2 \quad (23)$$

Solu<sup>n</sup>:- Given differential equ<sup>n</sup> is

$$(1+2x)^2 \frac{\partial^2 y}{\partial x^2} - 6(1+2x) \frac{\partial y}{\partial x} + 16y = 8(1+2x)^2$$

$$\text{put } 1+2x = e^z \Rightarrow z = \log(1+2x) \quad \text{& } b=2$$

let  $\frac{dy}{dz} = D$ .  $\therefore$  Equ<sup>n</sup> becomes

$$[2D^2 - (D-1) - 6(2D) + 16]y = 8(e^z)^2$$

$$[4(D^2 - D) - 12D + 16]y = 8(e^z)^2$$

$$[4(D^2 - 4D + 4)]y = 8(e^z)^2$$

$$4(D^2 - 4D + 4)y = 8(e^z)^2$$

$$(D^2 - 4D + 4)y = 2(e^z)^2$$

$$\therefore A.E \text{ is } m = 4m + 4 \text{ & } (m-2)^2 \neq 0, m=2, 2$$

$$C.F = C_1 + C_2 z e^{2z}$$

$$P.I = \frac{1}{(D-2)^2} 2e^{2z}$$

$$= 2z \frac{1}{(D-2)} e^{2z} \quad (\because f(D)=0)$$

$$z \cdot \frac{1}{(D-2)} e^{2z} = z^2 \frac{1}{1} e^{2z} \quad (\because f'(D)=0)$$

General Solu<sup>n</sup> is G.S = C.F + P.I

$$y = C_1 + C_2 z e^{2z} + z^2 e^{2z}$$

$$y = [C_1 + C_2 \log(1+2x)](e^z)^2 + \log(1+2x)^2 (e^z)^2$$

$$= [C_1 + C_2 \log(1+2x)](1+2x)^2 + \log(1+2x)^2 (1+2x)^2$$

$$(5) \text{ Solve } (3x+2)^2 \frac{d^2y}{dx^2} + 5(3x+2) \frac{dy}{dx} - 3y = x^2 + x + 1$$

Solu<sup>n</sup> - Given differential Equn is

$$\Rightarrow (3x+2)^2 \frac{d^2y}{dx^2} + 5(3x+2) \frac{dy}{dx} - 3y = x^2 + x + 1$$

$$\text{put } 3x+2 = e^z \quad \text{so} \quad z = \log(3x+2)$$

$$x = \frac{e^z - 2}{3} \quad \text{let } b = 3$$

Let  $\frac{d}{dz^2} = D \quad \therefore \text{Equn becomes}$

$$\Rightarrow [3^2 D(D-1) + 5(3D) - 3]y = \left(\frac{e^z - 2}{3}\right)^2 + \left(\frac{e^z - 2}{3}\right) + 1$$

$$\Rightarrow [9(D^2 - 1) + 15D - 3]y = \frac{e^{2z} - 4e^z + 4}{9} + \frac{e^z - 2}{3} + 1$$

$$\Rightarrow [9(D^2 - 9) + 15D - 3]y = \frac{e^{2z}}{9} - \frac{9}{4}e^z + 4 + 3e^z - 6 + 9$$

$$3[3D^2 + 2D - 1]y = \frac{1}{9}[e^{2z} - e^z + 7]$$

$$(3D^2 + 2D - 1)y = \frac{1}{27}(e^{2z} - e^z + 7)$$

$$\therefore A, B \text{ is } 3m^2 - 2m - 1 = 0$$

$$(m+1)(3m-1) = 0$$

$$m = -1, \frac{1}{3}$$

$$C.P. = c_1 e^{-2} + c_2 e^{\frac{1}{3}z}$$

$$P.I. = \frac{1}{3D^2 + 2D - 1} \times \frac{1}{27}(e^{2z} - e^z + 7)$$

$$= \frac{1}{27} \left[ \frac{1}{(3D^2 + 2D - 1)} e^{2z} - \frac{1}{(3D^2 + 2D - 1)} e^z + \frac{7}{3D^2 + 2D - 1} e^{0z} \right]$$

$$= \frac{1}{27} \left[ \frac{1}{3x^2 + 2x + 1} e^{2z} - \frac{1}{3x^2 + 2x + 1} e^z + \frac{7}{5x^2 + 1} e^{0z} \right]$$

$$P.I. = \frac{1}{27} \left[ \frac{e^{2z}}{15} - \frac{e^z}{4} - 7 \right]$$

$\therefore$  General Solu<sup>n</sup> is G.S. = R.F + P.I.

$$y = c_1 e^{-2} + c_2 e^{\frac{1}{3}z} + \frac{1}{27} \left[ \frac{e^{2z}}{15} - \frac{e^z}{4} - 7 \right]$$

$$y = c_1 (3x+2)^{-1} + c_2 (3x+2)^{\frac{1}{3}} + \frac{1}{27} \left[ \frac{(3x+2)^2}{15} - \frac{(3x+2)}{4} - 7 \right]$$

(29)

**Exact Differential Equations of  $n^{\text{th}}$  order**

The Differential Equations of  $n^{\text{th}}$  order is  
of the form

$$P_0 \frac{d^n y}{dx^n} + P_1 \frac{d^{n-1} y}{dx^{n-1}} + P_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + P_{n-1} \frac{dy}{dx} + P_n y = f(x) \quad (1)$$

is said to be exact when it can be derived from an equation of next lower order by differentiating only.

If without any further process the above equation can also be put as

$$F\left(\frac{d^n y}{dx^n}, \frac{d^{n-1} y}{dx^{n-1}}, \dots, \frac{dy}{dx}, y\right) = f(x)$$

**Conditions of Exactness for Linear Equns:-**

2017 P.T the condition for exactness of

$$2018: P_0 \frac{d^3 y}{dx^3} + P_1 \frac{d^2 y}{dx^2} + P_2 \frac{dy}{dx} + P_3 y = f(x)$$

where  $P_0, P_1, P_2, P_3$  are functions of  $x$ .

Solu<sup>n</sup>: Suppose the differential equation

$$P_0 \frac{d^3 y}{dx^3} + P_1 \frac{d^2 y}{dx^2} + P_2 \frac{dy}{dx} + P_3 y = f(x) \quad (1) \text{ is exact}$$

then by definition of exactness, the above equation must have been derived from the differential Equn of 2<sup>nd</sup> order

$$P_0 \frac{d^2 y}{dx^2} + Q_1 \frac{dy}{dx} + Q_2 y = \int f(x) dx + C \quad (2)$$

Dif<sup>t</sup> Equn (2) w.r.t  $x$

$$P_0 \frac{d^3 y}{dx^3} + P_0' \frac{d^2 y}{dx^2} + Q_1' \frac{dy}{dx} + Q_1 \frac{dy}{dx} + Q_2 \frac{dy}{dx} + Q_2' y = f(x)$$

$$\Rightarrow P_0 \frac{d^3 y}{dx^3} + (P_0' + Q_1) \frac{d^2 y}{dx^2} + (Q_1' + Q_2) \frac{dy}{dx} + Q_2' y = f(x) \quad (3)$$

Equn (1) & (3) represents same as they are differential of equn (2)

Now equating the coefficients of

$$\frac{d^3y}{dx^3}, \frac{d^2y}{dx^2}, \frac{dy}{dx}, y$$

$$\therefore P_0 = P_0, \quad P_1 = P_0' + Q_1 \\ \therefore Q_1 \Rightarrow P_1 - P_0' \Rightarrow Q_1 = P_1 - P_0''$$

$$P_2 = Q_1' + Q_2 \Rightarrow Q_2 = P_2 - Q_1'$$

$$Q_2 = P_2 - P_1' + P_0''$$

$$Q''' = P_2' - P_1'' + P_0'''$$

$$P_3 = Q_2' = P_2' - P_1'' + P_0'''$$

$$\therefore P_3 - P_2' + P_1' - P_0''' = 0 \quad \text{--- (4)}$$

. Eqn (4) is condition for exactness of (1)  
And this condition is satisfied then eqn (1)

is exact & its soln is

$$P_0 \frac{d^2y}{dx^2} + Q_1 \frac{dy}{dx} + Q_2 y = \int f(x) dx + C$$

$$\therefore e^{\int P_0 \frac{d^2y}{dx^2} + (P_1 - P_0') \frac{dy}{dx} + (P_2 - P_1' + P_0'')} y = \int f(x) dx + C$$

(2) Derive the condition for exactness of

$P_0 \frac{dy}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = f(x)$  is exact  
where  $P_0, P_1, P_2$  are functions of  $x$ .

Soln: Suppose the differential eqn

$P_0 \frac{dy}{dx^2} + P_1 \frac{dy}{dx} + P_2 y = f(x) \quad \text{(1)}$  is exact

By def<sup>n</sup> of exactness  $\int f(x) dx + C \quad \text{(2)}$

$$P_0 \frac{dy}{dx} + Q y = \int f(x) dx + C$$

Dif<sup>t</sup> Eqn (2) w.r.t  $x$

$$P_0 \frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + Q' y = f'(x) \quad \text{(3)}$$

$$P_0 \frac{d^2y}{dx^2} + (P_1 + Q') y = f'(x) \quad \text{(3)}$$

Eqn (1) & (3) are identical

$\therefore$  Equating coefficient of  $\frac{d^2y}{dx^2}, \frac{dy}{dx}, y$

$$P_0 = P_0$$

$$P_1 = P_1 + Q' \Rightarrow Q = P_1 - P_1 = Q' = P_1 - P_0$$

$$P_2 = Q' = P_1 - P_0$$

$$P_2 - P_1 + P_0 = 0 \quad \text{(4)}$$

which is required condition of exactness  
of eqn (1)

$$\therefore \boxed{P_0 \frac{dy}{dx} + (P_1 - P_0)y = \int f(x) dx + C}$$

(1) Verify the condition for exactness & then  
solve the equ<sup>n</sup>  $\frac{\partial^2 y}{\partial x^2} + 2x \frac{\partial y}{\partial x} + 2y = 0$

Sol<sup>n</sup>: Given differential equ<sup>n</sup> is  
 $x \frac{d^2 y}{dx^2} + 2x \frac{dy}{dx} + 2y = 0 \quad \text{--- (1)}$

$$P_0 = x, P_1 = 2x, P_2 = 2$$

The condition of exactness is

$$P_0 - P_1' + P_2'' = 2 - 2 + 0 = 0$$

Equ<sup>n</sup> (1) is exact & its integral is

$$P_0 \frac{dy}{dx} + (P_1 - P_0') y = \int P_2 dx + C$$

$$x \frac{dy}{dx} + (2x - 1)y = \int 0 dx + C$$

$$x \frac{dy}{dx} + (2x - 1)y = C$$

$$\frac{dy}{dx} + \frac{(2x - 1)}{x} y = \frac{C}{x}$$

which is linear in y

$$P = 2 - \frac{1}{x}, Q = \frac{C}{x}$$

$$\therefore I.F = e^{\int P dx} = e^{\int (2 - \frac{1}{x}) dx}$$

$$I.F = e^{2x - \log x} = e^{2x} \cdot e^{-\log x} = e^{2x} \cdot x^{-1} = e^{2x} \cdot \frac{1}{x}$$

$$\therefore \text{So lin is } y(I.F) = \int Q(I.F) dx$$

$$y \cdot e^{\int P dx} = \int \frac{C}{x} \times e^{2x} \times \frac{1}{x} dy$$

$$y \cdot e^{2x} = \int \frac{C}{x^2} e^{2x} dx + C$$

$$y \cdot e^{2x} = \int \frac{C}{x^2} e^{2x} dx + C$$

2018

Q. Verify the condition for exactness & then solve the eqn  $(1+x^2)\frac{\partial^2 y}{\partial x^2} + 3xy \frac{\partial y}{\partial x} + y = 0$  (3)

Solu<sup>n</sup>: Given differential eqn is  
 $(1+x^2)\frac{\partial^2 y}{\partial x^2} + 3xy \frac{\partial y}{\partial x} + y = 0 \quad \text{--- (1)}$

$$P_0 = 1+x^2, P_1 = 3xy, P_2 = 1$$

The condition of exactness is

$$P_0 - P_1 + P_2'' = 1 - 3 + 2 = 0$$

Eqn (1) is exact & its integral is

$$P_0 \frac{\partial y}{\partial x} + (P_1 - P_0)y = \int 0 dx + C_1$$

$$(1+x^2)\frac{\partial y}{\partial x} + (3xy - 2x)y = C_1$$

$$(1+x^2)\frac{\partial y}{\partial x} + xy = C_1$$

$$\frac{dy}{dx} + \left(\frac{x}{1+x^2}\right)y = \frac{C_1}{1+x^2}$$

which is linear in y

$$\therefore I.F = e^{\int P_1 dx} = e^{\int \frac{x}{1+x^2} dx} = e^{\frac{1}{2} \log(1+x^2)}$$

$$= e^{\sqrt{1+x^2}}$$

$\therefore$  The solu<sup>n</sup> is  $y(I.F) = \int Q(I.F) dx$

$$y \cdot \sqrt{1+x^2} = \int \frac{C_1}{1+x^2} \sqrt{1+x^2} dx$$

$$y \cdot \sqrt{1+x^2} = \int \frac{C_1}{\sqrt{1+x^2}} dx$$

$$y \cdot \sqrt{1+x^2} = C_1 \log \left[ x + \sqrt{x^2+1} \right] + C_2$$

$$= C_1 \underbrace{\log \left[ x + \sqrt{x^2+1} \right]}_{\sqrt{1+x^2}} + C_2 \underbrace{\frac{1}{\sqrt{1+x^2}}}_{\sqrt{1+x^2}}$$

3) Verify the condition for exactness & then  
Solve the equ<sup>n</sup>.  $\frac{\partial^2 y}{\partial x^2} + 2e^x \frac{\partial y}{\partial x} + 2e^{2x}y = x^2$

Solu<sup>n</sup>: Given differential equ<sup>n</sup> is

$$\frac{\partial^2 y}{\partial x^2} + 2e^x \frac{\partial y}{\partial x} + 2e^{2x}y = x^2 \quad \text{--- (1)}$$

$$\text{where } P_0 = 1, \quad P_1 = 2e^x, \quad P_2 = 2e^{2x}$$

The condition of exactness is

$$P_2 - P_1 + P_0'' = 2e^x - 2e^x + 0$$

∴ Equ<sup>n</sup> (1) is exact & its integral is

$$P_0 \frac{dy}{dx} + (P_1 - P_0')y = \int f(x)dx + C$$

$$\frac{dy}{dx} + (2e^x)y = \int x^2 dx + C$$

$$\frac{dy}{dx} + (2e^x)y = \frac{x^3}{3} + C_1$$

$$\therefore P = 2e^x, \quad Q = \frac{x^3}{3} + C_1$$

$$I.F = e^{\int P dx} = e^{\int 2e^x dx} = e^{2e^x}$$

∴ The solu<sup>n</sup> is

$$y \cdot (I.F) = \int Q(I.F) dx$$

$$y \cdot e^{2e^x} = \int \left( \frac{x^3}{3} + C_1 \right) e^{2e^x} dx + C_2$$

$$y \cdot e^{2e^x} = \int \left( \frac{x^3}{3} e^{2e^x} + e^{2e^x} C_1 \right) dx + C_2$$

(4) Verify the condition for exactness & the (32)  
 2016, 2017. Solve  $\sin x \frac{dy}{dx^2} - \cos x \frac{dy}{dx} + 2y \sin x = 0$   
 2018, 2019.

Solu<sup>n</sup>:- Given differential equ<sup>n</sup> is  
 $\sin x \frac{dy}{dx^2} - \cos x \frac{dy}{dx} + 2y \sin x = 0 \quad \text{--- (1)}$

where  $P_0 = \sin x$ ,  $P_1 = -\cos x$ ,  $P_2 = 2y \sin x$

The condition of exactness is

$$P_2 - P_1' + P_0'' = 2\sin x - \sin x - \sin x = 0$$

∴ The equ<sup>n</sup> (1) is exact & solu<sup>n</sup> is

$$\Rightarrow P_0 \frac{dy}{dx} + (P_1 - P_0') y = \int f(x) dx + C$$

$$\Rightarrow \sin x \frac{dy}{dx} + (-\cos x - \cos x) y = \int 0 dx + C$$

$$\Rightarrow \sin x \frac{dy}{dx} - 2\cos x y = C, \quad \text{--- (2)}$$

$$\frac{dy}{dx} - 2 \left( \frac{\cos x}{\sin x} \right) y = \frac{C}{\sin x}$$

where  $P = -2 \cot x$ ,  $Q = \frac{C}{\sin x}$

$$I.F = e^{\int P dx} = e^{\int -2 \cot x dx} = e^{\log(\sin x)} \\ = e^{\log(\sin x)^{-2}}$$

$$I.F = (\sin x)^2$$

∴ Solu<sup>n</sup> is  $y(I.F) = \int Q I.F dx$

$$y \cdot \frac{1}{(\sin x)^2} = \int \frac{C}{\sin x} \cdot \frac{1}{(\sin x)^2} dx$$

$$y \cdot (\csc x)^2 = \int C \csc x \csc^2 x dx$$

$$y \cdot \csc^2 x = \int C \csc^2 x dx$$

$$\text{put } \cot x = t$$

$$-\csc^2 x dx = dt$$

$$y \csc^2 x = \int \sqrt{1+t^2} dt$$

$$\therefore \csc^2 x - \cot^2 x = 1$$

$$= -\int \sqrt{1+t^2} dt$$

$$y \cdot \csc^2 x = -\left[ \frac{t}{2} \sqrt{t^2+1} + \frac{1}{2} \log(t + \sqrt{t^2+1}) \right]$$

$$y \cdot \csc^2 x = -\left[ \frac{\cot u}{2} \sqrt{\cot^2 x + 1} + \frac{1}{2} \log(\cot x + \sqrt{1+\cot^2 x}) \right]$$

5) Verify the condition for exactness & solve

$$x \frac{d^3y}{dx^3} + (x^2-3) \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = 0$$

Soln: Given D.E is  $x \frac{d^3y}{dx^3} + (x^2-3) \frac{d^2y}{dx^2} + 4x \frac{dy}{dx} + 2y = 0$

$$\text{where } P_0 = x, P_1 = x^2-3, P_2 = 4x, P_3 = 2$$

The condition of exactness is

$$P_3 - P_2' + P_1'' - P_0''' = 2 - 4 + 2 = 0 \neq 0$$

$\therefore$  The equ<sup>n</sup> (1) is exact & its integral is

$$P_0 \frac{dy}{dx} + (P_1 - P_0') y = \int f(x) dx + C_1$$

$$x \frac{dy}{dx} + (x^2-3-1) y = \int (4x-2x+0) dy = C_1$$

$$x \frac{dy}{dx} + (x^2-4) y = C_1 \quad \dots (2)$$

$$\text{Again } P_0 = x, P_1 = x^2-4, P_2 = 2x$$

The condition of exactness is

$$P_2 - P_1' + P_0''' = 0 - 2/x + 2x = 0$$

$\therefore$  equ<sup>n</sup> (2) is exact & its integral is

$$P_0 \frac{dy}{dx} + (P_1 - P_0') y = \int C_1 dx + C_2$$

$$x \frac{dy}{dx} + (x^2-4-1) y = C_1 x + C_2$$

$$x \frac{dy}{dx} + (x^2-5) y = C_1 x + C_2$$

$$\frac{dy}{dx} + \frac{(x^2-5)}{x} y = C_1 x + C_2$$

which is linear in y

$$I.F = e^{\int x^2-5 dx} = e^{\int x^2-5 x dx} = e^{x^2/2 - 5 \log x}$$

$$I.F = e^{x^2/2} \cdot y = \int e^{x^2/2} (C_1 x + C_2) dx$$

$$y \cdot e^{x^2/2} = \int e^{x^2/2} (C_1 x + C_2) dx$$

**B.Sc. IV SEM. UNIT IV DIFFERENTIAL EQUATIONS**

**Differential Equations:-** I) Type-3(P.I.for RHS=  $x^n$ ),  
 II) Type-4(P.I.for RHS=  $e^x \sin x$ ), III) Type-5(P.I.for RHS=  $xv$ )

**Type 3**

**1. Solve**  $y'' + 3y' + 2y = 12x^2$

We have  $(D^2 + 3D + 2)y = 12x^2$

A.E is  $m^2 + 3m + 2 = 0$  or  $(m+1)(m+2) = 0 \Rightarrow m = -1, -2$

$$y_c = c_1 e^{-x} + c_2 e^{-2x}$$

$$y_p = \frac{12x^2}{D^2 + 3D + 2}$$

We need to divide for obtaining the P.I.

$$\begin{array}{r} 6x^2 - 18x + 21 \\ \hline 2 + 3D + D^2 \end{array} \left| \begin{array}{r} 12x^2 \\ 12x^2 + 36x + 12 \\ \hline -36x - 12 \\ -36x - 54 \\ \hline 42 \\ \frac{42}{0} \end{array} \right. \left\{ \begin{array}{l} \text{Note: } 3D(6x^2) = 3 \frac{d}{dx}(6x^2) = 3 \cdot 12x = 36x \\ D^2(6x^2) = 12 \end{array} \right\}$$

Hence  $y_p = 6x^2 - 18x + 21$

Complete solution:  $y = y_c + y_p$

Thus  $y = c_1 e^{-x} + c_2 e^{-2x} + 6x^2 - 18x + 21$

**2. Solve**  $y'' + 2y' + y = 2x + x^2$

We have  $(D^2 + 2D + 1)y = 2x + x^2$

A.E. is  $m^2 + 2m + 1 = 0 \Rightarrow (m+1)^2 = 0 \Rightarrow m = -1, -1$

$$\therefore y_c = (c_1 + c_2 x)e^{-x}$$

$$y_p = \frac{2x+x^2}{D^2+2D+1} = \frac{x^2+2x}{D^2+2D+1} \text{ P.I. is found by division}$$

$$\begin{array}{r} x^2 - 2x + 2 \\ 1 + 2D + D^2 \left| \begin{array}{r} x^2 + 2x \\ x^2 + 4x + 2 \\ \hline -2x - 2 \\ -2x - 4 \\ \hline 2 \\ 2 \\ \hline 0 \end{array} \right. \\ \therefore y_p = x^2 - 2x + 2 \end{array}$$

Complete Solution:  $y = y_c + y_p$ . Thus  $y = (c_1 + c_2 x)e^{-x} + (x^2 - 2x + 2)$

---

**3. Solve**  $(D^3 + 8)y = x^4 + 2x + 1$

A.E. is  $m^3 + 8 = 0 \Rightarrow (m+2)(m^2 - 2m + 4) = 0$

$$m = -2 \text{ and } m = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm 2i\sqrt{3}}{2} = 1 \pm i\sqrt{3}$$

$$\therefore y_c = c_1 e^{-2x} + e^x (c_2 \cos \sqrt{3}x + c_3 \sin \sqrt{3}x)$$

$$y_p = \frac{x^4 + 2x + 1}{8 + D^3} \text{ P.I. is found by division}$$

$$8 + D^3 \left| \begin{array}{r} x^4/8 - x/8 + 1/8 \\ x^4 + 2x + 1 \\ \hline x^4 + 3x \\ -x + 1 \\ \hline -x - 0 \\ \hline 1 \\ \hline 1 \\ \hline 0 \end{array} \right.$$

Here  $D^3(x^4/8) = 3x$

$$\therefore y_p = \frac{1}{8}(x^4 - x + 1)$$

Complete Solution:  $y = y_c + y_p$

**Thus**  $y = c_1 e^{-2x} + e^x \{c_2 \cos \sqrt{3}x + c_3 \sin \sqrt{3}x\} + \frac{1}{8}(x^4 - x + 1)$

---

**4. Solve**  $\frac{d^3x}{dt^3} + 3\frac{d^2x}{dt^2} = 1 + t$

We have  $(D^3 + 3D^2)x = 1 + t$  where  $D = \frac{d}{dt}$

A.E. is  $m^3 + 3m^2 = 0 \Rightarrow m^2(m+3) = 0 \Rightarrow m = 0, 0, -3$

$$\therefore x_c = c_1 + c_2 t + c_3 e^{-3t}$$

$$\therefore y_c = c_1 e^{-2x} + e^x (c_2 \cos \sqrt{3}x + c_3 \sin \sqrt{3}x)$$

$$x_p = \frac{1+t}{D^3 + 3D^2} = \frac{1+t}{3D^2 + D^3}$$

P.I. is found by division

$$3D^2 + D^3 \quad t^3/18 - t^2/9$$

$$\begin{array}{c} \boxed{t+1} \\ \boxed{t+1/3} \\ \hline 2/3 \\ \hline 2/3 \\ \hline 0 \end{array} \quad \begin{aligned} \text{Here } \frac{1}{3D^2}(t) &= \frac{1}{3} \int \int t dt dt = \frac{1}{3} \cdot \frac{t^3}{6} = \frac{t^3}{18} \\ D^3 \left( \frac{t^3}{18} \right) &= \frac{1}{3} \quad \& \quad \frac{2/3}{3D^2} = \frac{2}{9} \int \int dt dt = \frac{t^2}{9} \end{aligned}$$

$$\text{Hence } x_p = \frac{t^3}{18} + \frac{t^2}{9} = \frac{t^2}{9}(t+2) \quad \text{Complete Solution: } x = x_c + x_p$$

$$\text{Thus } x = c_1 + c_2 t + c_3 e^{-3t} + \frac{t^2(t+2)}{18}$$


---

**5. Solve**  $\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 6y = x^2$

We have  $(D^2 + 5D + 6)y = x^2$

A.E.      is       $m^2 + 5m + 6 = 0 \Rightarrow (m+2)(m+3) = 0 \Rightarrow m = -2, -3$   
 $\therefore y_c = c_1 e^{-2x} + c_2 e^{-3x}$

$y_p = \frac{x^2}{6+5D+D^2}$  P.I. is found by division

$$x^2/6 - 5x/18 + 19/108$$

$$\begin{array}{c} 6+5D+D^2 \quad \boxed{x^2} \\ \hline x^2(5x/3)++(1/3) \\ \hline -(5x/3)-(1/3) \\ \hline -(5x/3)-(25/18) \\ \hline 19/18 \\ \hline 19/18 \\ \hline 0 \end{array} \quad \begin{aligned} \text{Here } 5D(x^2/6) &= 5x/3 \\ D^2(x^2/6) &= 1/3 \quad \& \quad 5D(5x/18) \\ &= -25/18 \\ \therefore y_p &= \frac{1}{108}(18x^2 - 30x + 19) \end{aligned}$$

Complete Solution:  $y = y_c + y_p$

$$\text{Thus } y = c_1 e^{-2x} + c_2 e^{-3x} + \frac{1}{108} (18x^2 - 30x + 19)$$


---

**6. Solve**  $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^3$

We have  $(D^3 + 2D^2 + D)y = x^3$

$$\text{A.E. is } m^3 + 2m^2 + m = 0 \Rightarrow m(m^2 + 2m + 1) = m(m+1)^2 = 0$$

$$\Rightarrow m = 0, -1, -1 \quad \therefore y_c = c_1 + (c_2 + c_3 x)e^{-x}$$

$$y_p = \frac{x^3}{D + 2D^2 + D^3} \text{ P.I. is found by division}$$

$$\begin{array}{r} x^4/4 - 2x^3 + 9x^2 - 24x \\ \hline D + 2D^2 + D^3 \left| \begin{array}{r} x^3 \\ x^3 + 6x^2 + 6x \\ -6x^2 - 6x \\ -6x^2 - 24x - 12 \\ 18x + 12 \\ 18x + 36 \\ -24 \\ -24 \\ 0 \end{array} \right. \end{array} \quad \begin{array}{l} \text{Here } \frac{x^3}{D} = \int x^3 dx = \frac{x^4}{4}, \\ 2D^2(x^4/4) \\ = D^2(x^4/2) = 6x^2, \\ \frac{-6x^2}{D} = \int -6x^2 dx = -2x^3 \\ \frac{18x}{D} = \int 18x dx = 9x^2 \\ \& \frac{-24}{D} = -24x \end{array}$$

$$\therefore y_p = \frac{x^4}{4} - 2x^3 + 9x^2 - 24x \text{ Complete Solution: } y = y_c + y_p$$

---


$$\text{Thus } y = c_1 + (c_2 + c_3 x)e^{-x} + \frac{x^4}{4} - 2x^3 + 9x^2 - 24x$$

#### Type-4

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1. Solve  $(D^2 - 2D + 5)y = e^{2x} \sin x$

A.E. is  $m^2 - 2m + 5 = 0 \Rightarrow m = 1 \pm 2i$

$$\therefore y_c = e^x (c_1 \cos 2x + c_2 \sin 2x)$$

$$\begin{aligned} y_p &= \frac{e^{2x} \sin x}{D^2 - 2D + 5} = e^{2x} \left[ \frac{\sin x}{(D+2)^2 - 2(D+2) + 5} \right] \left( \begin{array}{l} \text{since replace } D \\ \text{by } D+2 \end{array} \right) \\ &= e^{2x} \left[ \frac{\sin x}{D^2 + 2D + 5} \right] = \frac{e^{2x}}{2} \left[ \frac{\sin x}{D+2} \right] \left( \because D^2 \rightarrow -1^2 = -1 \right) \\ &= \frac{e^{2x}}{2} \left[ \frac{(D-2)\sin x}{(D-2)(D+2)} \right] = \frac{e^{2x}}{2} \cdot \frac{\cos x - 2\sin x}{D^2 - 4} = \frac{e^{2x}}{2} \cdot \frac{\cos x - 2\sin x}{-1-4} \\ y_p &= \frac{e^{2x}}{10} \cdot (2\sin x - \cos x) \end{aligned}$$

Complete Solution:  $y = y_c + y_p$

Thus  $y = e^x (c_1 \cos 2x + c_2 \sin 2x) + \frac{e^{2x}}{10} \cdot (2\sin x - \cos x)$

---

2. Solve  $(D^2 - 2D + 4)y = e^x \cos x$

A.E. is  $m^2 - 2m + 4 = 0 \Rightarrow m = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm 2i\sqrt{3}}{2} = 1 \pm i\sqrt{3}$

$$\therefore y_c = e^x (c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x)$$

$$\begin{aligned} y_p &= \frac{e^x \cos x}{D^2 - 2D + 4} = e^x \left[ \frac{\cos x}{(D+1)^2 - 2(D+1) + 4} \right] \left( \begin{array}{l} \text{since replace } D \\ \text{by } D+1 \end{array} \right) \\ &= e^x \left[ \frac{\cos x}{D^2 + 3} \right] = e^x \left[ \frac{\cos x}{2} \right] \left( \because D^2 \rightarrow -1^2 = -1 \right) \quad \therefore y_p = \frac{e^x \cos x}{2} \end{aligned}$$

Complete Solution:  $y = y_c + y_p$

$$\text{Thus } y = e^x \left( c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x \right) + \left( e^x \cos x \right) / 2$$


---

**3. Solve**  $(D^3 + 1)y = 5e^x x^2$

A.E. is  $m^3 + 1 = 0 \Rightarrow (m+1)(m^2 - m + 1) = 0$

$$\Rightarrow m = -1, \quad m = \frac{1 \pm \sqrt{1-4}}{2} = \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

$$\therefore y_c = c_1 e^{-x} + e^{x/2} \left\{ c_2 \cos \left( \sqrt{3}/2 \right)x + c_3 \sin \left( \sqrt{3}/2 \right)x \right\}$$

$$\begin{aligned} y_p &= \frac{5e^x x^2}{D^3 + 1} = 5e^x \left[ \frac{x^2}{(D+1)^3 + 1} \right] \left( \begin{array}{l} \text{since replace } D \\ \text{by } D+1 \end{array} \right) \\ &= 5e^x \left[ \frac{x^2}{D^3 + 3D^2 + 3D + 2} \right] = \frac{5e^x}{2} \left[ \frac{2x^2}{2 + 3D + 3D^2 + D^3} \right] \left( \begin{array}{l} \text{since for a convenient} \\ \text{division} \times \& \div \text{by 2} \end{array} \right) \end{aligned}$$

$$\begin{array}{c} x^2 - 3x + (3/2) \\ \hline 2 + 3D + 3D^2 + D^3 \left| \begin{array}{r} 2x^2 \\ 2x^2 + 6x + 6 \\ - 6x - 6 \\ \hline - 6x - 9 \\ 3 \\ \hline 3 \\ \hline 0 \end{array} \right. \end{array} \quad \begin{aligned} \therefore y_p &= \frac{5e^x}{2} \cdot [x^2 - 3x + (3/2)] \\ &= (5e^x/4) \cdot [2x^2 - 6x + 3] \end{aligned}$$

Complete Solution:  $y = y_c + y_p$

$$\begin{aligned} \text{Thus } y &= c_1 e^{-x} + e^{x/2} \left\{ c_2 \cos \left( \sqrt{3}/2 \right)x + c_3 \sin \left( \sqrt{3}/2 \right)x \right\} \\ &\quad + (5e^x/4) \cdot [2x^2 - 6x + 3] \end{aligned}$$


---

**4. Solve**  $(D^2 - 4D + 3)y = 2xe^{3x}$

A.E. is  $m^2 - 4m + 3 = 0 \Rightarrow (m-1)(m-3) = 0 \Rightarrow m = 1, 3$

$$\therefore y_c = c_1 e^x + c_2 e^{3x}$$

$$y_p = \frac{2e^{3x}x}{(D-1)(D-3)} = 2e^{3x} \left[ \frac{x}{(D+2)D} \right] \begin{matrix} (\because \text{replace } D) \\ (\text{by } D+3) \end{matrix}$$

$$= e^{3x} \cdot \frac{2x}{D^2 + 2D} \quad \text{P. I. is found by division}$$

$$(x^2/2) - (x/2)$$

$$2D + D^2 \left| \begin{array}{r} 2x \\ \hline 2x+1 \\ -1 \\ \hline -1 \\ \hline 0 \end{array} \right. \quad \because \frac{2x}{2D} = \int x dx = \frac{x^2}{2} \quad \therefore y_p = (x^2/2) - (x/2)$$

Complete Solution:  $y = y_c + y_p$

Thus  $y = c_1 e^x + c_2 e^{3x} + (x^2/2) - (x/2)$

---

**5. Solve**  $x''(t) - 4x(t) = t \sinh t$

We have  $(D^2 - 4)x(t) = t \sinh t$  where  $D = \frac{d}{dt}$

A.E. is  $m^2 - 4 = 0 \Rightarrow m = \pm 2 \quad \therefore x_c = c_1 e^{2t} + c_2 e^{-2t}$

$$x_p = \frac{t \sinh t}{D^2 - 4} = \frac{1}{2} \frac{t \cdot (e^t - e^{-t})}{D^2 - 4} = \frac{1}{2} \left[ \frac{t \cdot e^t}{D^2 - 4} - \frac{t \cdot e^{-t}}{D^2 - 4} \right] = \frac{1}{2} [P_1 - P_2] \quad (\text{say})$$

$$P_1 = \frac{t \cdot e^t}{D^2 - 4} = e^t \frac{t}{(D+1)^2 - 4} = e^t \frac{t}{D^2 + 2D - 3} \quad \text{P. I. is found by division}$$

$$-3 + 2D + D^2 \begin{vmatrix} t \\ \frac{t - (2/3)}{2/3} \\ \frac{2/3}{0} \end{vmatrix} \quad \therefore P_1 = \left( \frac{-t}{3} - \frac{2}{9} \right) e^t$$

$$P_2 = \frac{t \cdot e^{-t}}{D^2 - 4} = e^{-t} \frac{t}{(D-1)^2 - 4} = e^{-t} \frac{t}{D^2 - 2D - 3}$$

$$-3 - 2D + D^2 \begin{vmatrix} t \\ \frac{t + (2/3)}{-2/3} \\ \frac{-2/3}{0} \end{vmatrix} \quad \therefore P_2 = \left( \frac{-t}{3} + \frac{2}{9} \right) e^{-t} \quad \therefore x_p = \frac{1}{2} [P_1 - P_2]$$

$$= \frac{1}{2} \left[ -\left( \frac{t}{3} + \frac{2}{9} \right) e^t - \left( \frac{-t}{3} + \frac{2}{9} \right) e^{-t} \right]$$

$$= \frac{-t}{3} \sinh t - \frac{2}{9} \cosh t$$

Complete Solution:  $x = x_c + x_p$

$$\text{Thus } x = c_1 e^{2t} + c_2 e^{-2t} - \frac{t}{3} \sinh t - \frac{2 \cosh t}{9}$$

**6. Solve**  $\frac{d^4 y}{dt^4} - y = e^t \cos t$

We have  $(D^4 - 1)y = e^t \cos t$ , where  $D = \frac{d}{dt}$

$$\text{A.E. is } m^4 - 1 = 0 \Rightarrow (m^2 - 1)(m^2 + 1) = 0 \Rightarrow m = 1, -1, \pm i$$

$$\therefore y_c = c_1 e^t + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$$

$$\begin{aligned}
y_p &= \frac{e^t \cos t}{D^4 - 1} = e^t \cdot \frac{\cos t}{(D+1)^4 - 1} = e^t \cdot \frac{\cos t}{D^4 + 4D^3 + 6D^2 + 4D + 1 - 1} \\
&= e^t \cdot \frac{\cos t}{D^4 + 4D^3 + 6D^2 + 4D}, \quad \text{Now } D^2 \rightarrow -1 \\
&= e^t \cdot \frac{\cos t}{(-1)^2 + 4(-1)D + 6(-1)^2 + 4D} = e^t \cdot \frac{\cos t}{-5}
\end{aligned}$$

Complete Solution:  $y = y_c + y_p$

$$\text{Thus } y = c_1 + (c_2 + c_2 x) e^{-x} + \frac{x^4}{4} - 2x^3 + 9x^2 - 24x$$


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### Type-5

1. Solve  $y'' + 16y = x \sin 3x$

We have  $(D^2 + 16)y = x \sin 3x$ . A.E. is  $m^2 + 16 = 0 \Rightarrow m = \sqrt{-16} = \pm 4i$

$$\therefore y_c = c_1 \cos 4x + c_2 \sin 4x. \quad y_p = \frac{x \sin 3x}{D^2 + 16}.$$

$$\begin{aligned}
\text{Let us use } \frac{xV}{f(D)} &= \left[ x - \frac{f'(D)}{f(D)} \right] \frac{V}{f(D)} = \left[ x - \frac{2D}{D^2 + 16} \right] \frac{\sin 3x}{(D^2 + 16)} \\
&= \left[ x - \frac{2D}{D^2 + 16} \right] \frac{\sin 3x}{(-3^2 + 16)} = \frac{x \sin 3x}{7} - \frac{6 \cos 3x}{7(D^2 + 16)} = \frac{x \sin 3x}{7} - \frac{6 \cos 3x}{7(-3^2 + 16)} \\
y_p &= \frac{x \sin 3x}{7} - \frac{6 \cos 3x}{49} = \frac{1}{49} [7x \sin 3x - 6 \cos 3x]
\end{aligned}$$

Complete Solution:  $y = y_c + y_p$ .

$$\text{Thus } y = c_1 \cos 4x + c_2 \sin 4x + \frac{1}{49} [7x \sin 3x - 6 \cos 3x]$$

---

**2. Solve**  $y'' - 2y' + y = x \cos x$

We have  $(D^2 - 2D + 1)y = x \cos x$ . A.E. is  $m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$

$$\therefore y_c = (c_1 + c_2 x)e^x. \quad y_p = \frac{x \cos x}{D^2 - 2D + 1}.$$

Let us use

$$\begin{aligned} \frac{xV}{f(D)} &= \left[ x - \frac{f'(D)}{f(D)} \right] \frac{V}{f(D)} = \left[ x - \frac{2D - 2}{D^2 - 2D + 1} \right] \frac{\cos x}{(D^2 - 2D + 1)} \\ &= \left[ x - \frac{2(D-1)}{(D-1)^2} \right] \frac{\cos x}{(-1 - 2D + 1)} = \left[ x - \frac{2(D-1)}{(D-1)^2} \right] \frac{\sin x}{(-2)} \quad \left\{ \because \frac{\cos x}{(-2D)} = \int \frac{\cos x}{(-2)} dx = \frac{\sin x}{(-2)} \right\} \\ &= \frac{-x \sin x}{2} + \frac{\sin x}{(D-1)} = \frac{-x \sin x}{2} + \frac{(D+1) \sin x}{D^2 - 1} = \frac{-x \sin x}{2} + \frac{\cos x + \sin x}{-2} \end{aligned}$$

Complete Solution:  $y = y_c + y_p$ .

$$\text{Thus } y = (c_1 + c_2 x)e^x - \frac{1}{2}(x \sin x + \cos x + \sin x)$$


---

**3. Solve**  $y'' - y = x^2 \cos x$

We have  $(D^2 - 1)y = x^2 \cos x$ .

$$\text{A.E. is } m^2 - 1 = 0 \Rightarrow m = \pm 1 \quad \therefore y_c = c_1 e^x + c_2 e^{-x}$$

$$y_p = \frac{x^2 \cos x}{D^2 - 1} = \frac{R.P.(x^2 e^{ix})}{D^2 - 1} \quad \left\{ \because \text{do not prefer to apply the result } \frac{xV}{f(D)} \text{ twice} \right\}$$

$$y_p = R.P.e^{ix} \frac{x^2}{D^2 - 1}, \quad \{ \text{replace } D \text{ by } D + i \}$$

$$= R.P.e^{ix} \frac{x^2}{(D+i)^2 - 1} = R.P.e^{ix} \frac{x^2}{D^2 + 2iD + i^2 - 1}$$

$$= R.P.e^{ix} \frac{x^2}{D^2 + 2iD - 2} = R.P.e^{ix} \frac{x^2}{-2 + 2iD + D^2}.$$

We need to employ division

$$\begin{array}{r}
 \left( x^2/-2 \right) - ix + (1/2) \\
 \overline{-2 + 2iD + D^2} \quad \left| \begin{array}{l} 2x \\ x^2 - 2ix - 1 \\ 2ix + 1 \\ \hline 2ix + 2 \\ -1 \\ -1 \\ \hline 0 \end{array} \right. \quad \begin{array}{l} \therefore 2iD(x^2/2) = -2ix \\ D^2(x^2/2) = -1 \\ \text{Quotient is} \\ \frac{-1}{2}(x^2 - 1) - ix \end{array}
 \end{array}$$

$$\begin{aligned}
 y_p &= R.P e^{ix} \left[ \frac{1}{2}(1-x^2) - ix \right] = R.P (\cos x + i \sin x) \left[ \frac{1}{2}(1-x^2) - ix \right] \\
 &= R.P \left\{ \left[ \frac{1}{2} \cos x (1-x^2) + x \sin x \right] + i \left[ \frac{1}{2} \sin x (1-x^2) - x \cos x \right] \right\}
 \end{aligned}$$

Required  $y_p = \frac{1}{2} \cos x (1-x^2) + x \sin x$ . Complete Solution:  $y = y_c + y_p$ .

---


$$\text{Thus } y = c_1 e^x + c_2 e^{-x} + \frac{\cos x}{2} (1-x^2) + x \sin x$$

---

#### 4. Solve $y'' - y = x^2 \sin x$

We have  $(D^2 - 1)y = x^2 \sin x$ . The working is same as in the above example

where  $y_p = I.P. \frac{x^2 e^{ix}}{D^2 - 1}$ . Required  $y_p = \frac{\sin x}{2} (1-x^2) - x \cos x$ .

Complete Solution:  $y = y_c + y_p$ .

---


$$\text{Thus } y = c_1 e^x + c_2 e^{-x} + \frac{\sin x}{2} (1-x^2) - x \cos x$$

---

#### 5. Solve $y'' - 2y' + y = xe^x \sin x$

We have  $(D^2 - 2D + 1)y = xe^x \sin x$ . A.E. is  $m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$   
 $\therefore y_c = (c_1 + c_2 x)e^x$ .

$$y_p = e^x \frac{x \sin x}{(D-1)^2} \text{ First } D \rightarrow D+1 \quad \therefore y_p = e^x \left[ \frac{x \sin x}{D^2} \right]$$

And we have

$$\begin{aligned} \frac{xV}{f(D)} &= \left[ x - \frac{f'(D)}{f(D)} \right] \frac{V}{f(D)} \\ y_p &= e^x \left[ x - \frac{2D}{D^2} \right] \frac{\sin x}{D^2} = e^x \left[ x - \frac{2}{D} \right] \frac{\sin x}{-1} = e^x (-x \sin x + 2 \cdot -\cos x) \end{aligned}$$

$y_p = -e^x (x \sin x + 2 \cos x)$  Complete Solution:  $y = y_c + y_p$   
 Complete Solution:  $y = y_c + y_p$ .

$$\text{Thus } y = (c_1 + c_2 x)e^x - \frac{1}{2}(x \sin x + \cos x + \sin x)$$

The working is same as in the above example where  $y_p = I.P. \frac{x^2 e^{ix}}{D^2 - 1}$ .

$$\text{Required } y_p = \frac{\sin x}{2} (1 - x^2) - x \cos x.$$

Complete Solution:  $y = y_c + y_p$ .

$$\text{Thus } y = c_1 e^x + c_2 e^{-x} + \frac{\sin x}{2} (1 - x^2) - x \cos x$$

**6. Solve**  $y'' - 4y' + 4y = 8x^2 e^{2x} \cos 2x$

We have  $(D^2 - 4D + 4)y = 8x^2 e^{2x} \cos 2x$ .

$$\text{A.E. is } m^2 - 4m + 4 = 0 \Rightarrow m = 2, 2 \quad \therefore y_c = (c_1 + c_2 x)e^{2x}.$$

$$y_p = \frac{8x^2 e^{2x} \cos 2x}{(D-2)^2} = e^{2x} \frac{8x^2 \cdot \cos 2x}{(D-1)^2} \text{ First } D \rightarrow D+2 \quad \therefore y_p = e^{2x} \left[ \frac{8x^2 \cdot \cos 2x}{D^2} \right]$$

$$y_p = \left(e^{2x}\right) \frac{R.P(8x^2 e^{2ix})}{D^2} \left\{ \begin{array}{l} \Leftrightarrow P.I \text{ can also be computed by} \\ \text{integrating } 8x^2 \cos 2x \text{ twice by parts} \end{array} \right\}$$

$$= \left(e^{2x}\right) R.P e^{2ix} \left[ \frac{8x^2}{D^2} \right] = \left(e^{2x}\right) R.P e^{2ix} \frac{8x^2}{(D+2i)^2} = \left(e^{2x}\right) R.P e^{2ix} \frac{8x^2}{D^2 + 4iD - 4}$$

We need to employ division

$$\begin{array}{r} -2x^2 - 4ix + 3 \\ \hline -4 + 4iD + D^2 \left| \begin{array}{r} 8x^2 \\ 8x^2 - 16ix - 4 \\ \hline 16ix + 4 \\ 16ix + 16 \\ \hline -12 \\ -12 \\ \hline 0 \end{array} \right. \end{array}$$

Quotient is  $-2x^2 - 4ix + 3$

$$\begin{aligned} y_p &= R.P e^{2ix} \left[ -2x^2 - 4ix + 3 \right] \\ &= e^{2x} R.P (\cos 2x + i \sin 2x) \left[ -2x^2 - 4ix + 3 \right] \\ &= e^{2x} \left\{ R.P (\cos 2x + i \sin 2x) \left[ -2x^2 - 4ix + 3 \right] \right\} \\ &= e^{2x} \left\{ \cos 2x (3 - 2x^2) + 4x \sin 2x \right\} \end{aligned}$$

Complete Solution:  $y = y_c + y_p$ .

Thus  $y = (c_1 + c_2 x) e^{2x} + e^{2x} \left\{ \cos 2x (3 - 2x^2) + 4x \sin 2x \right\}$

**7. Solve**  $y'' - 4y' + 4y = 8x^2 e^{2x} \sin 2x$  *{refer above example}*

We have  $(D^2 - 4D + 4)y = 8x^2 e^{2x} \sin 2x$ .

$$\begin{aligned}
y_p &= I.P e^{2ix} \left[ -2x^2 - 4ix + 3 \right] \\
&= e^{2x} I.P (\cos 2x + i \sin 2x) \left[ -2x^2 - 4ix + 3 \right] \\
&= e^{2x} \left\{ \sin 2x (3 - 2x^2) - 4x \cos 2x \right\}
\end{aligned}$$

Complete Solution:  $y = y_c + y_p$ .

$$\text{Thus } y = (c_1 + c_2 x) e^{2x} + e^{2x} \left\{ \sin 2x (3 - 2x^2) - 4x \cos 2x \right\}$$


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#### Type-6. Mixed type of examples

**1. Solve**  $(D^3 + 1)y = \cos(\pi/2 - x) + e^x$

$$\begin{aligned}
\text{A.E. is } m^3 + 1 = 0 &\Rightarrow (m+1)(m^2 - m + 1) = 0 \\
\Rightarrow m = -1, \quad m = \frac{1 \pm \sqrt{1-4}}{2} &= \frac{1}{2} \pm \frac{i\sqrt{3}}{2} \\
\therefore y_c &= c_1 e^{-x} + e^{x/2} \left\{ c_2 \cos(\sqrt{3}/2)x + c_3 \sin(\sqrt{3}/2)x \right\} \\
y_p &= \frac{\sin x + e^x}{D^3 + 1} = \frac{\sin x}{D^3 + 1} + \frac{e^x}{D^3 + 1} = p_1 + p_2 \quad \left( \begin{array}{l} \text{since replace } D \\ \text{by } D+1 \end{array} \right) \\
p_1 &= \frac{\sin x}{D^3 + 1} \text{ now } D^2 \rightarrow -1^2 = -1 \\
\therefore p_1 &= \frac{\sin x}{1-D} = \frac{(1+D)\sin x}{(1-D)(1+D)} = \frac{\sin x + \cos x}{1-D^2} = \frac{\sin x + \cos x}{2} \\
p_2 &= \frac{e^x}{D^3 + 1} \text{ Here } D \rightarrow 1 \therefore p_2 = \frac{e^x}{2}
\end{aligned}$$

Complete Solution:  $y = y_c + y_p$ .

$$\therefore y_c = c_1 e^{-x} + e^{x/2} \left\{ c_2 \cos(\sqrt{3}/2)x + c_3 \sin(\sqrt{3}/2)x \right\} + \frac{\sin x + \cos x + e^x}{2}$$


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**2. Solve**  $\frac{d^2y}{dx^2} - 2 \frac{dy}{dx} + y = xe^x + x$

We have  $(D^2 - 2D + 1)y = xe^x + x$

A.E. is  $m^2 - 2m + 1 = 0 \Rightarrow (m-1)^2 = 0 \Rightarrow m = 1, 1,$

$$\therefore y_c = (c_1 + c_2 x)e^x \text{ and } y_p = \frac{xe^x}{D^2 - 2D + 1} + \frac{x}{D^2 - 2D + 1} = p_1 + p_2 \text{ (say)}$$

$$p_1 = e^x \frac{x}{(D-1)^2}; D \rightarrow D+1 \quad \therefore p_1 = e^x \frac{x}{D^2} = e^x \int \int x dx dx = e^x \frac{x^3}{6} = \frac{x^3 e^x}{6}$$

$$p_2 = \frac{x}{1-2D+D^2} \text{ and we shall divide}$$

$$\begin{array}{r} x+2 \\ 1-2D+D^2 \left| \begin{array}{r} x \\ x-2 \\ \hline 2 \\ \hline 2 \\ \hline 0 \end{array} \right. \end{array} \quad \therefore p_2 = x+2$$

*Complete solution:  $y = y_c + y_p$*

*Thus  $y = (c_1 + c_2 x)e^x + x^3 e^x / 6 + (x+3)$*

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**3. Solve**  $y'' + 4y' - 12y = e^{2x} - 3\sin 2x$

We have  $(D^2 + 4D - 12)y = e^{2x} - 3\sin 2x.$

A.E. is  $m^2 + 4m - 12 = 0 \Rightarrow (m+6)(m-2) = 0 \Rightarrow m = 2, -6$

$$\therefore y_c = (c_1 + c_2 x)e^{2x}.$$

$$y_p = \frac{e^{2x}}{(D^2 + 4D - 12)} - \frac{3\sin 2x}{(D^2 + 4D - 12)} = p_1 + p_2 \text{ (say)}$$

$$p_1 = \frac{e^{2x}}{(D^2 + 4D - 12)} = \frac{e^{2x}}{(4+8-12)} \quad (Dr. = 0) \quad \therefore p_1 = x \frac{e^{2x}}{2D+4} = \frac{xe^{2x}}{8}$$

$$p_2 = \frac{-3\sin 2x}{(D^2 + 4D - 12)}; D^2 \rightarrow -4 \quad \therefore p_2 = \frac{-3\sin 2x}{(-4+4D-12)} = \frac{-3(D+4)\sin 2x}{4(D^2-16)}$$

$$p_2 = \frac{-3(2\cos 2x + 4\sin 2x)}{-80} \quad \text{Complete Solution: } y = y_c + y_p$$

$$\text{Thus } y = c_1 e^{2x} + c_2 e^{-6x} + \frac{xe^{2x}}{8} + \frac{3(\cos 2x + 2\sin 2x)}{40}$$

---

**4. Solve**  $\frac{d^2y}{dx^2} - y = (1+x^2)e^x + x \sin x$

We have  $(D^2 - 1)y = (1+x^2)e^x + x \sin x$

A.E. is  $m^2 - 1 = 0 \Rightarrow m = \pm 1 \quad \therefore y_c = c_1 e^x + c_2 e^{-x}$  and

$$y_p = \frac{(1+x^2)e^x}{D^2 - 1} + \frac{x \sin x}{D^2 - 1} = p_1 + p_2 \text{ (say)}$$

$$p_1 = \frac{(1+x^2)e^x}{D^2 - 1} = e^x \frac{1+x^2}{(D+1)^2 - 1} \quad (\because D \rightarrow D+1) \quad \therefore p_1 = e^x \frac{x^2 + 1}{2D + D^2}$$

We need to employ division now

$$\begin{array}{r|l} & x^3/6 - x^2/4 + 3x/4 \\ \hline 2D + D^2 & \overline{x^2 + 1} \\ & \overline{x^2 + x} \\ & \overline{-x + 1} \\ & \overline{-x - 1/2} \\ & \overline{3/2} \\ & \overline{3/2} \\ & 0 \end{array} \quad \begin{array}{l} \frac{x^2}{2D} = \int \frac{x^2}{2} dx = \frac{x^3}{6} \\ \frac{-x}{2D} = \int \frac{-x}{2} dx = \frac{-x^2}{4} \\ \frac{3/2}{2D} = \frac{3}{4} \int dx = \frac{3x}{4} \end{array}$$

$$\therefore \text{quotient} = \frac{x^3}{6} - \frac{x^2}{4} + \frac{3x}{4} = \frac{x}{12}(2x^2 - 3x + 9) \quad \therefore p_1 = e^x \cdot \frac{x}{12}(2x^2 - 3x + 9)$$

And we have

$$\begin{aligned} p_2 &= \frac{x \sin x}{D^2 - 1} \text{ and we use } \frac{xV}{f(D)} = \left[ x - \frac{f'(D)}{f(D)} \right] \frac{V}{f(D)} \\ p_2 &= \left[ x - \frac{2D}{D^2 - 1} \right] \frac{x \sin x}{D^2 - 1} = \left[ x - \frac{2D}{D^2 - 1} \right] \frac{x \sin x}{-1 - 1} = \frac{-x \sin x}{2} + \frac{\cos x}{D^2 - 1}; \\ \text{As } D^2 &\rightarrow -1, p_2 = \frac{-x \sin x}{2} + \frac{\cos x}{-2} = \frac{-1}{2}(x \sin x + \cos x) \end{aligned}$$

Complete Solution:  $y = y_c + y_p$ .

$$\text{Thus } y = c_1 e^x + c_2 e^{-x} - \frac{x e^x}{12} (2x^2 - 3x + 9) - \frac{1}{2} (x \sin x + \cos x)$$


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**5. Solve**  $(D^3 + D^2 - 4D - 4)y = 3e^{-x} - 4x - 6$

A.E. is  $m^3 + m^2 - 4m - 4 = 0 \Rightarrow m^2(m+1) - 4(m+1) = 0$

$$\Rightarrow (m+1)(m^2 - 4) = 0 \Rightarrow m = -1, \pm 2 \quad \therefore y_c = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{-2x}$$

$$y_p = \frac{3e^{-x}}{(D^3 + D^2 - 4D - 4)} - \frac{(4x+6)}{(D^3 + D^2 - 4D - 4)} = p_1 - p_2 \text{ (say)}$$

$$p_1 = \frac{3e^{-x}}{(D^3 + D^2 - 4D - 4)} = \frac{3e^{-x}}{-1+1+4-4} \text{ (Dr. = 0)}$$

$$p_1 = x \cdot \frac{3e^{-x}}{(3D^2 + 2D - 4)} = x \cdot \frac{3e^{-x}}{(3-2-4)} = -xe^{-x}$$

$$p_2 = \frac{(4x+6)}{(D^3 + D^2 - 4D - 4)} \quad \text{P.I. is found by division}$$

$$-x - (1/2)$$

$$\begin{array}{r} -4 - 4D + D^2 + D^3 \\ \hline 4x + 6 \\ 4x + 4 \\ \hline 2 \\ \hline 2 \\ \hline 0 \end{array} \quad \therefore p_2 = -x - (1/2)$$

*Complete solution:  $y = y_c + y_p$*

$$\text{Thus } y = c_1 e^{-x} + c_2 e^{2x} + c_3 e^{-2x} - xe^{-x} + x + (1/2)$$


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**6. Solve**  $\frac{d^2y}{dt^2} + 4y = t \sin t + \sin 2t$

We have  $(D^2 + 4)y = t \sin t + \sin 2t$

A.E. is  $m^2 + 4 = 0 \Rightarrow m = \pm 2i \quad \therefore y_c = c_1 \cos 2t + c_2 \sin 2t$  and

$$y_p = \frac{t \sin t + \sin 2t}{D^2 + 4} = \frac{t \sin t}{D^2 + 4} + \frac{\sin 2t}{D^2 + 4} = p_1 + p_2 \text{ (say)}$$

$$\begin{aligned}
p_1 &= \frac{t \sin t}{D^2 + 4} \text{ and we have } \frac{xV}{f(D)} = \left[ t - \frac{f'(D)}{f(D)} \right] \frac{V}{f(D)} \\
&= \left[ t - \frac{2D}{D^2 + 4} \right] \frac{\sin t}{D^2 + 4} = \left[ t - \frac{2D}{D^2 + 4} \right] \frac{\sin t}{-1^2 + 4} = \frac{t \sin t}{3} - \frac{2 \cos t}{3(D^2 + 4)} \\
\therefore p_1 &= \frac{t \sin t}{3} - \frac{2 \cos t}{9} = \frac{1}{9}(3t \sin t - 2 \cos t)
\end{aligned}$$

Also  $p_2 = \frac{\sin 2t}{D^2 + 4}$  Now  $D^2 \rightarrow -2^2$  and  $Dy = 0$

$$p_2 = t \frac{\sin 2t}{2D} = t \int \frac{\sin 2t}{2} dt = -\frac{t \cos 2t}{4}$$

Complete Solution:  $y = y_c + y_p$ .

$$\text{Thus } y = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{9}(3t \sin t - 2 \cos t) - \frac{t \cos 2t}{4}$$